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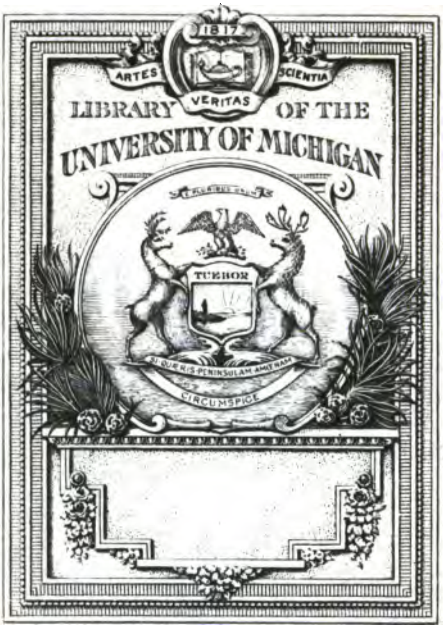
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A NEW AND COMPLETE
TREATISE
OF
SPHERICAL
TRIGONOMETRY:

In which are contained the

ORTHOGRAPHIC, ANALYTICAL and LOGARITHMICAL

SOLUTIONS

OF THE SEVERAL CASES OF

SPHERICAL TRIANGLES,

Whether right-angled or oblique;

A COMPREHENSIVE THEORY of the FLUXIONS of theſe TRIANGLES;

AND

A Variety of CURIOUS and INTERESTING Particulars not to be met with in any other TREATISE upon this Subject.

CAREFULLY TRANSLATED FROM THE FRENCH OF

MR. MAUDUIT,

By W. CRAKELT.

LONDON,
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MDCCLXVIII.

TO

JOHN BEVIS, M. D.

FELLOW OF THE ROYAL SOCIETY,

AND

OF THE ROYAL ACADEMY OF SCIENCES
AT BERLIN,

THIS TRANSLATION

IS INSCRIBED BY

HIS MUCH OBLIGED

AND OBEDIENT SERVANT,

W. CRAKELT.

PREFACE.

MOST of the Treatiſes of ſpherical Trigonometry, that have been hitherto publiſhed, contain only directions for ſolving ſpherical Triangles *(Logarithmically)* by proportions; and none, that I know of, hath united the different ſolutions whereof Problems in this part of the Mathematics are ſuſceptible.——Now, with a very little attention to the nature of the ſubject, it will appear that the ſolutions may be of three kinds, viz. 1°, ſuch as reſpect *Analogies* (of which kind are thoſe given in all the common Treatiſes of ſpherical Trigonometry); 2°, ſuch as are obtained by the ſcale and compaſs from *Projections* or *Geometrical Conſtructions*, and 3°, ſuch as are deduced from projections by the application of *Algebra*; all which will be treated of at large in the following Work.

In the firſt Chapter I have laid down the definitions and fundamental principles of *ſines*, *coſines*, *tangents*, *cotangents*, &c. in a great variety of particular formulæ, as well as infinite ſeries. I have likewiſe inſerted a ſhort Theory upon the *true* nature of *Imaginaries* (ſince they have been frequently uſed in ſeries relative to the different powers of *ſines* and *coſines* of multiple arcs), and, from an attempt to diſcover the Factors of the binomial $aa + bb$, have eaſily inferred, that theſe expreſſions are only ſigns of an abſurd ſuppoſition into which we neceſſarily fall, whenever we regard that as the product of two quantities which in reality is not. But, for what I have ſaid upon this head, I acknowledge myſelf indebted to the late ingenious, and much regretted, Mr. *Boiſgelou*.

Some may perhaps conſider this part of the Work as a diſplay of algebraical formulæ, entirely uſeleſs in a Treatiſe of Trigonometry: however, from ſuch I ſhall take the liberty of requeſting a ſuſpenſion of judgment, at leaſt till they have undertaken the ſtudy of the different works of calculation relative to the theory of *Newton*: for Aſtronomy is become ſo excellent and delicate a ſcience, that it is impoſſible to have too many helps for facilitating the calculations it requires, and perfecting the ſeveral parts upon which it depends; and for thoſe Gentlemen, who have no intention of ſtudying this ſubject thoroughly, a peruſal of the five or ſix firſt Theorems may be ſufficient.

In the ſecond Chapter I have demonſtrated the principal properties of ſpherical triangles; premiſing, in the firſt of the four Sections into which this Chapter is divided, the neceſſary definitions, &c. The ſecond Section

tion treats of the reſolution of right-angled ſpherical triangles; and, as this part is of almoſt perpetual uſe, I have attempted to give it the greateſt ſimplicity whereof it can be ſuſceptible; for every Theorem requires only a ſingle analogy for its demonſtration: and, to enable the memory ſtill more to retain the different ſolutions, I have added a demonſtration of the general Theorem of *Neper*, which reduces the ſixteen caſes of right-angled triangles to two, and requires only the bare enumeration of theſe caſes to be verified. The third Section contains the reſolution of all the caſes of oblique-angled ſpherical triangles. Herein I have inſerted a Theorem analogous to the Theorem of *Neper* for right-angled triangles, and the ſame with that which Mr. *Pingré* hath given in the *Memoirs of the Academy*, except for a few trifling alterations, which I thought neceſſary to be made in order to render it ſomewhat more ſimple. I have likewiſe ſuperadded to the ſolutions already given, for the caſe of three ſides or three angles, ſeveral others which I take to be new; and which, when applied to right-lined triangles, furniſh us with ſolutions much more eaſy and compendious than any to be met with in common Treatiſes upon Trigonometry. In the laſt Section complete demonſtrations of the famous Analogies or Theorems of *Nepar* (which have been conſiderably mutilated and altered in the Courſe of Mr. *Wolff* are given; and ſuch as carry along with them ſeveral other Theorems equally general, whereof Mr. *Simpſon* hath ſpecified only particular caſes. To this Section I have annexed three Tables containing the ſubſtance of the whole Work, at leaſt as far as the more ordinary practice extends, and have expreſſed them in words inſtead of letters, that miſapprehenſions and miſtakes in calculations might be avoided as much as poſſible. The firſt Table is for the reſolutions of right-angled ſpherical triangles, and the ſecond and third for thoſe of oblique-angled ones; but the laſt is, by means of the analogies of *Neper*, equally applicable to right-lined triangles, and may moreover have this advantage attending it; that, on account of the perfect reſemblance prevailing between the ſolutions of ſimilar caſes of theſe two kinds of triangles, a perſon, previouſly acquainted with the proportions for plain triangles, may learn the principal parts of ſpherical Trigonometry in a quarter of an hour. ——As this Chapter contains all that is eſſential and neceſſary to the reſolution of ſpherical triangles, they, who would make themſelves maſters of this part of Trigonometry only, may paſs from hence immediately to the ſixth Chapter, in order to ſee the method of putting its formulæ into Logarithms.

The third Chapter contains the Geometrical or Graphical ſolutions of the different caſes that had been before ſolved by analogies. I have, in preference to the reſt, made uſe of the *orthographic* projections, whereof Aſtronomers ſuppoſe at every inſtant a complete theory: however, I have ſaid ſomething upon the *ſtereographic* projections alſo, on account of the great and frequent uſe which is made of them in Maps; and have, laſtly, ſubjoined a new ſolution eaſily applicable to the ſeveral caſes before ſpecified, and deduced from the expanſion of the parts of the triangle under conſideration.

The fourth Chapter contains an application of the Algebraical analyſis to the Geometrical conſtructions in the preceding Chapter. In this part it is that, by the help of calculation, the ſolutions for all particular caſes are (if I may be allowed the expreſſion) exhauſted. Moſt of the analogies of the ſecond Chapter are deduced from general caſes in this by ſimple inferences: a variety of new formulæ is added to thoſe that had been before given; and the apparent difference between the ſynthetic and analytic ſolutions for the ſame caſes perfectly reconciled. And, that nothing might be wanting, I have ſhewn the method of conſtructing the moſt complicated of the formulæ by the Logarithms: after which I have given the ſolutions of equations of the ſecond and third degree by the Tables of *ſines*, &c. a thing that may prove to be of the greateſt utility, when commenſurable roots cannot be obtained. The reſolutions of equations of the third degree were communicated to me by a gentleman already well known in the world by different works, and yet leſs eſtimable on account of his profound learning than the excellence of his character: to him I likewiſe owe the new ſolution of the Problem concerning the ſhorteſt twilight, inſerted in the fifth Chapter.

Which is little more than a Tranſlation of that excellent Treatiſe of Mr. *Cotes*: "De *æſtimatione errorum in mixtâ Matheſi*," containing an application of the modern analyſis or the calculation of Fluxions * to both plain and ſpherical Trigonometry. What relates to right-lined triangles I have deduced from the formulæ for ſpherical triangles, that I might render the Theory as ſimple as poſſible, and abridge in ſome meaſure a Work that was already become much longer than I at firſt intended; and, in the laſt place, have ſufficiently elucidated this Theory by the application of it to different examples.

In the ſixth Chapter I have ſhewn the method of conſtructing by the Logarithms the formulæ given in the ſecond Chapter; and this I have done in a ſelect collection of Aſtronomical Problems, through a perſuaſion that ſolutions derived from triangles having ſome particular denominations, could not but prove much more intereſting to Learners, than ſuch as were derived from triangles abſtractedly conſidered.

---

* In this Chapter the Tranſlator has taken the liberty of changing the *Differential* method of conſideration into the *Fluxional*, and alſo of ſupplying ſuch references and compariſons as appeared to him neceſſary and conducive to the more ſpeedy comprehenſion of ſeveral of its parts.

The motive which firſt induced me to undertake the writing of this Treatiſe aroſe from hence: Having had occaſion to teach the Elements of Aſtronomy of the late Abbé *De la Caille*, I found, upon a peruſal of the little Treatiſe of Trigonometry prefixed to that Work, a great number of formulæ without demonſtrations, and therefore, well knowing that Learners are generally deſirous of making themſelves perfect as they proceed in the ſtudy of Mathematical truths, I thought that I ſhould do them no inconſiderable piece of ſervice in giving them all requiſite aſſiſtance in ſuch difficulties as few would otherwiſe be able to ſurpaſs. How well I have ſucceeded in my deſign, as alſo in the execution of the other parts of the Work, I leave to the Public to determine: however, ſhould it be found deſerving of their encouragement, it ſhall be followed by another Treatiſe upon Dialling as its ſequel. 'Tis particularly in this part of the Mathematics that ſpherical Trigonometry is advantageouſly applied: for the deſcriptions of Dials by lines are not ſufficiently exact to deſerve any certain confidence; and there hath been yet no Treatiſe upon Dialling publiſhed (that I know of) wherein the whole hath been performed by the calculation of ſpherical triangles, though ſuch method could not fail to prove of the utmoſt importance in the practice, as all the different hour-angles, which are uſually calculated by a ſeries of plain triangles, might be then determined without the leaſt relation to one another.

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ERRATA.

Page 89, line 8, for *half the ſum* read *the ſum*; p. 104, l. 11, for *d*D read *d*G; p. 135, l. 16, for 112 read 212; p. 137, laſt line, for $\frac{f}{mr}$ read $\frac{fgh}{mr}$; p. 158, l. 27, for *ſine* read *ſign*; p. 186, l. 28, for *ſide* read *angle*.

THE PRINCIPLES OF SPHERICAL TRIGONOMETRY.

CHAP. I.

Containing the Definitions and chief Properties of the ſeveral lines which are uſually conſidered in Trigonometry; but particularly of ſuch as we ſhall have more frequent occaſion to mention in the courſe of this Work.

DEFINITIONS.

ART. I. A Right line M P, drawn from one end of an arc A M, or angle A C M, perpendicularly to the radius which paſſeth through the other end thereof, is called the *Sine* of ſuch arc or angle. *Fig.* 1.

2. From this definition it is evident, that the right line M Q is the *ſine* of the arc B M, the

complement of the arc AM; but if this line be conſidered with relation to the arc AM, it is called its *Coſine.*

3. A right line AT perpendicular to the end of a radius, and terminated by the radius produced which paſſeth through the other end M of an arc AM, is called the *Tangent* of this arc.

4. A right line B*t*, which is the *tangent* of the arc BM, the complement of AM, when conſidered with relation to the ſaid arc AM, is called its *Cotangent.*

5. The line intercepted between the centre of the circle and the *tangent* of an arch, is called the *Secant* of that arch: thus CT is the *ſecant* of the arch AM, and C*t* that of the arch BM: but if the *ſecant* C*t* be conſidered with relation to the arch AM, the complement of the arch BM whereunto it belongs, it will be called the *Coſecant* thereof.

6. That part of the radius contained between the circumference and *ſine* of an arc, is called the *Verſed-ſine* of ſuch arc: thus AP is the *verſed-ſine* of the arc AM; BQ that of the arc BM; and *a*P and *b*Q the *verſed-ſines* of the obtuſe angles *a*CM, *b*CM. If the *verſed-ſine* BQ be conſidered as belonging to an arc which is the complement of AM, with reſpect to the ſaid arch AM, it will be called the *Coverſed-ſine*; and contrarily, AP will be the *coverſed ſine* of the arch BM.

COROLLARY.

7. From theſe definitions it follows, that two angles, which are the ſupplements to each other, have the ſame *ſine*, *coſine*, *tangent*, *cotangent*, *ſecant*, *coſecant*, and *verſed-ſines*. It muſt be obſerved however, that the obtuſe angle has for its complement the angle of its exceſs above 90°; which angle being evidently the ſame with that which the acute angle wants of 90°, muſt of conſequence be regarded as negative.

ADVERTISEMENT.

8. In all the following parts of this work we ſhall denote the radius of the circle by R or *r*, unleſs we ſuppoſe it equal to unity; which ought ſeldom to be done, that the homogeneity of the terms in the calculations may be preſerved as much as poſſible. We ſhall alſo denote the *ſine* of an arch by *ſin.* its *coſine* by *coſ.* its *tangent* by *tang.* its *cotangent* by *cot.* its *ſecant* by *ſec.* its *coſecant* by *coſec.* its *verſed-ſine* by *verſ-ſin.* and its *coverſed-ſine* by *cover-ſin:* but in the algebraical expreſſions we ſhall put the *ſine* $=s$; *coſine* $=c$; *tangent* $=t$; *cotangent* $=\tau$; *ſecant* $=S$; *coſecant* $=\int$; *verſed-ſine* $=v$, and *coverſed-ſine* $=u$.

THEOREM I.

9. *Let any arch of a circle, as* AM, *be aſſumed, (which we ſhall always denote by* A), *whereof* PM *is the ſine, and* MQ *the coſine*; AT *the tangent, and* B*t* *the cotangent*; *and the coſine of this arch will be to the ſine, as radius is to the tangent*; that is, *Coſ.* A : *ſin.* A : : R : *tang.* A. Fig. 1.

DEMONSTRATION.

The ſimilar triangles CPM and CAT give CP : PM : : CA : AT; or, *Coſ.* A : *ſin.* A : : R : *tang.* A. Q. E. D.

COROLLARY I.

10. The triangles CQM, CB*t*, being alſo ſimilar, will give CQ : QM : : CB : B*t*; or, *Sin.* A : *coſ.* A : : R : *cot.* A.

COROLLARY II.

11. It follows from hence, that *tang.* A $= \frac{\text{ſin. A}}{\text{coſ. A}} \times$ R, and *cot.* A $= \frac{\text{coſ. A}}{\text{ſin. A}} \times$ R; or algebraically, $t = \frac{sr}{c}$, and $\tau = \frac{cr}{s}$; and conſequently, *ſin.* A $\times$ R $=$ *coſ.* A $\times$ *tang.* A, and *coſ.* A $= \frac{\text{ſin. A} \times \text{cot. A}}{\text{R}}$.

COROLLARY III.

12. Hence it again follows, that the *cotangents* of any number of arcs are always in the inverſe ratio of the *tangents* of theſe arcs; ſince we have evidently, $t : \tau :: \frac{sr}{c} : \frac{cr}{s} :: \frac{s}{c} : \frac{c}{s}$ *: in ſhort, *tang.* A will be $= \frac{RR}{cot\ A}$; or, *tang.* A × *cot.* A = R R; whence we may perceive, that the radius is a mean proportional between the *tangent* of an arch and that of its complement; a thing which might likewiſe be eaſily demonſtrated by means of the ſimilar triangles CAT, CB*t* †.

THEOREM II.

13. *Let the arch* AM *be ſtill retained, and we ſhall* Fig. 2. *alſo have this proportion; As radius is to twice the ſine of the arch* AM, *ſo is the coſine of this arch to the ſine of twice the arch*; or, which is the ſame, R : 2 *ſin.* A : : *coſ.* A : *ſin.* 2A; or *alternando*, R : *coſ.* A : : 2 *ſin.* A : *ſin.* 2 A.

DEMONSTRATION.

The right-angled triangles APC, AQN, having the angle at A common, are ſimilar, and give AC : CP : : AN *or* 2 PA : QN; that is to ſay, R : *coſ.* A : : 2 *ſin.* A : *ſin.* 2 A. Q. E. D.

If in like manner we would find an expreſſion for the *coſine* of the double arch; the ſimilar triangles CPA, CQO, will give CA : CP : : CO : CQ; from whence (after putting for CO its value drawn from CP—OP or its equal PT [*]), we ſhall immediately get, $CQ = \frac{2\,\textit{coſ.}^2 A - RR}{R}$.

---

* *Or thus.* Since *cot.* A $= \frac{RR}{tang.\ A}$, and *cot.* B $= \frac{RR}{tang.\ B}$; therefore we ſhall have, *Cot.* A : *cot.* B : : $\frac{RR}{tang.\ A} : \frac{RR}{tang.\ B}$: : *tang.* B : *tang.* A.

† For AT : CA : : CB : B*t*; or, *tang.* A : R : : R : *cot.* A; wherefore, *tang.* A × *cot.* A = R R.

COROLLARY.

14. It follows from hence (*viz.* by taking the product of the extremes and means in the proportion, R : *coſ.* A : : 2 *ſin.* A : *ſin.* 2 A,) that, $\frac{1}{2}\textit{ſin.}\ 2A = \frac{\textit{coſ.}\ A \times \textit{ſin.}\ A}{R}$; and if in this equation we put $\frac{\textit{ſin.}\ A \times \textit{cot.}\ A}{R}$ for *coſ.* A, we ſhall otherwiſe have, $\frac{1}{2}\textit{ſin.}\ 2A = \frac{\textit{ſin.}^2 A \times \textit{cot.}\ A}{RR}$; or, $\frac{1}{2}\textit{ſin.}\ 2A = \frac{\textit{ſin.}^2 A}{\textit{tang.}\ A}$, by ſubſtituting for the *cotangent* its value in the *tangent*.

THEOREM III.

Fig. 3.

15. *The radius is a mean proportional*; 1°, *between the coſine of an arch and its ſecant*; *and* 2°, *between the ſine of the ſame arch and its coſecant*; and therefore we ſhall have theſe two analogies; *Coſ.* A : R : : R : *ſec.* A, and *Sin.* A : R : : R : *coſec.* A.

DEMONSTRATION.

The ſimilar triangles CPM, CAT, and CB*t* give 1°, CP : CA : : CM : CT; or, *Coſ.* A : R : : R : *ſec.* A; and 2°, PM : CM : : CB : C*t*; or, *Sin.* A : R : : R : *coſec.* A. Q.E.D.

---

[*] That OP is = PT, may be very eaſily proved: for by letting fall the perpendicular PS, it will appear that ſince AP = PN, therefore QS = SA, and conſequently OP = PT. This being granted, the value of CO will be obtained as follows: Since the angle CAT is right, PT or OP will be equal $\frac{\overline{PA}^2}{CP} = \frac{\overline{CA}^2 - \overline{CP}^2}{CP}$, and CO $= CP - \frac{\overline{CA}^2 - \overline{CP}^2}{CP} = \textit{coſ.}\ A - \frac{RR - \textit{coſ.}^2 A}{\textit{coſ.}\ A} = \frac{2\,\textit{coſ.}^2 A - RR}{\textit{coſ.}\ A}$.

COROLLARY I.

16. It follows from hence, that $\frac{\text{ſec. A}}{\text{coſec. A}} \times R = \text{tang. A}$; for we have already ſeen in *art.* 11 that *tang.* $A = \frac{\text{ſin. A}}{\text{coſ. A}} \times R$; and this quantity is by the preſent Theorem equal to the quotient $\frac{\text{ſec. A}}{\text{coſec. A}}$ multiplied by the radius.

COROLLARY II.

17. It moreover follows from hence, that the *ſines* of any two arcs, A and B, will be reciprocally proportional to the *coſecants* of theſe arcs; and the *coſines* to their *ſecants*. For ſince, *coſec.* $A = \frac{RR}{\text{ſin. A}}$, *coſec.* B will alſo be $= \frac{RR}{\text{ſin. B}}$; likewiſe, ſince *ſec.* $A = \frac{RR}{\text{coſ. A}}$, we ſhall have *ſec.* $B = \frac{RR}{\text{coſ. B}}$, and conſequently theſe two analogies; *Coſec.*A : *coſec.*B :: $\frac{RR}{\text{ſin. A}} : \frac{RR}{\text{ſin. B}}$ :: *ſin.* B : *ſin.* A; and, *Sec.* A : *ſec.* B :: $\frac{RR}{\text{coſ. A}} : \frac{RR}{\text{coſ. B}}$ :: *coſ.* B : *coſ.* A.

THEOREM IV.

Fig. 3.

18. *Let the arch* AM *and its complement* BM *be divided into two equal parts at the points* K *and* k, *and let the lines* CKl, CkL, *be drawn till they meet the tangents* AT *and* Bt *produced, as far as neceſſary, in* l *and* L; and we ſhall by this means have, 1°, *Sec.* A = *cot.* ½ *comp.* A — *tang.* A, and 2°, *Coſec.* A = *cot.* ½ A — *cot.* A.

DEMONSTRATION.

The right-angled triangles CB*o* and CAL being ſimilar, on account of the parallels B*o* and CA, will have the angles ALC, BC*o*, equal; but BC*o* is = *o*CM, by conſtruction; wherefore the triangle CTL is iſoſceles, and conſequently LT = CT;

but $LT = AL - AT$; $AL = \mathit{cot.}\ \frac{1}{2}\ \mathit{comp.}\ AM$, and $AT = \mathit{tang.}\ AM$; and therefore CT, or $\mathit{Sec.}\ A = \mathit{cot.}\ \frac{1}{2}\ \mathit{comp.}\ A - \mathit{tang.}\ A$. Q. E. 1°. D.

2. The right-angled triangles CAO and CB*l* being ſimilar, on account of the parallels CA and B*l*, will have the angles B*l*C and ACO equal; but ACO is = MCO, by conſtruction; therefore the triangle C*tl* is iſoſceles, and the *coſecant* $Ct = tl$: but it is evident that $tl = Bl - Bt = \mathit{cot.}\ \frac{1}{2} A - \mathit{cot.}\ A$; and conſequently the *Coſecant* C*t* of the arch AM is equal to the ſame quantity. Q. E. 2°. D.

THEOREM V.

19. *From the point* M *to the extremities* b *and* a *of the diameters* Bb, Aa, *let the lines* Mb, Ma, *be drawn, cutting the radii* CA *and* CB *in the points* G *and* g; *moreover let there be drawn through the ſame point* M *the tangent* $\theta M \tau$, *terminated by the radii* CA *and* CB *produced, as far as neceſſary, at* θ *and* τ, *and by that means determining the lines* $C\theta$ *and* $C\tau$ *reſpectively equal to the ſecant and coſecant of the arch* AM; then we ſhall have, 1°, $\mathit{Sec.}\ A = \mathit{tang.}\ A + \mathit{tang.}\ \frac{1}{2}\ \mathit{comp.}\ A$, and 2°, $\mathit{Coſec.}\ A = \mathit{cot.}\ A + \mathit{tang.}\ \frac{1}{2} A$. *Fig.* 3.

DEMONSTRATION.

The angle $bM\theta$, which is formed by the *tangent* $M\theta$ and chord M*b*, is meaſured by half the arch MA*b* compriſed between its ſides; alſo the angle $MG\theta$, which hath its vertex within the circumference, is meaſured by half the ſum of the arcs *ab*, AM, contained between its ſides; but *ab* is = A*b*, and therefore the angle $MG\theta$ equal to the angle $GM\theta$; conſequently the triangle $G\theta M$ is iſoſceles, and $G\theta = M\theta = \mathit{tang.}\ A$. Moreover, the angle G*b*C, which is at the circumference, is only half the angle BCM, which hath its vertex at the centre and ſtands upon the ſame arc; therefore, if we regard the radius C*b* as the ſine total, CG will be the *tangent* of half the complement of the arch A M; and conſequently, $\textit{Sec.}\ C\theta = CG + G\theta = \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A + \textit{tang.}\ A$†. Q. E. 1°. D.

2°. We might prove exactly in the ſame manner that, $g\tau = M\tau = \textit{cot.}\ A$, and $Cg = \textit{tang.}\ \tfrac{1}{2}A$; but it is evident that $C\tau = \textit{coſec.}\ A$; and therefore we ſhall have, $\textit{Coſec.}\ A = \textit{cot.}\ A + \textit{tang.}\ \tfrac{1}{2}A$. Q. E. 2°. D.

COROLLARY.

20. By comparing the two expreſſions of the *ſecants* and *coſecants* of the arch A M in the two preceding Theorems, it will follow; 1°. that, $\textit{cot.}\ \tfrac{1}{2}\ \textit{comp.}\ A - \textit{tang.}\ A = \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A + \textit{tang.}\ A$; and 2°. that, $\textit{cot.}\ \tfrac{1}{2}A - \textit{cot.}\ A = \textit{cot.}\ A + \textit{tang.}\ \tfrac{1}{2}A$: conſequently we ſhall likewiſe have, $2\ \textit{tang.}\ A = \textit{cot.}\ \tfrac{1}{2}\ \textit{comp.}\ A - \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A$; and, $2\ \textit{cot.}\ A = \textit{cot.}\ \tfrac{1}{2}A - \textit{tang.}\ \tfrac{1}{2}A$; and if for the *cot.* in the ſecond member of each of theſe equations there be ſubſtituted its value $\frac{RR}{\textit{tang.}}$ (*art.* 12), we ſhall again have, $\textit{tang.}\ A = \frac{RR - \textit{tang.}^2\ \frac{1}{2}\ \textit{comp.}\ A}{2\ \textit{tang.}\ \frac{1}{2}\ \textit{comp.}\ A}$; and, $\textit{cot.}\ A = \frac{RR - \textit{tang.}^2\ \frac{1}{2}A}{2\ \textit{tang.}\ \frac{1}{2}A}$; from whence the two following analogies may be eaſily deduced:

1°, As $2\ \textit{Tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A : R + \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A :: R - \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A : \textit{tang.}\ A$; or, which is the ſame, $2\ \textit{tang.}\ \overline{45^\circ - \tfrac{1}{2}A} : R + \textit{tang.}\ \overline{45^\circ - \tfrac{1}{2}A} :: R - \textit{tang.}\ \overline{45^\circ - \tfrac{1}{2}A} : \textit{tang.}\ A$; and 2°, as $2\ \textit{Tang.}\ \tfrac{1}{2}A : R + \textit{tang.}\ \tfrac{1}{2}A :: R - \textit{tang.}\ \tfrac{1}{2}A : \textit{cot.}\ A$.

THEOREM

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† *Otherwiſe thus.* Becauſe in the right-angled triangle C M o, the angle CoM is the complement of the angle oCM, or its equal oCB, and likewiſe the angle oCθ is the complement of the ſame angle oCB to the right angle θCB; therefore the angles θoC and θCo will be equal, and conſequently θC, or $\textit{Sec.}\ A = \theta a = \theta M + Mo = \textit{tang.}\ A + \textit{tang.}\ \tfrac{1}{2}\ \textit{comp.}\ A$.

THEOREM VI.

21. *Let there ſtill be an arch* AM *deſcribed with a radius* CA; and we ſhall have (by making the radius unity); 1°, $1+\textit{coſ.}\,A = 2\,\textit{coſ.}^2\,\frac{1}{2}A = \frac{\textit{ſin.}\,A\times R}{\textit{tang.}\,\frac{1}{2}A}$; and 2°, $1-\textit{coſ.}\,A = 2\,\textit{ſin.}^2\,\frac{1}{2}A = \textit{ſin.}\,A\times\textit{tang.}\,\frac{1}{2}A$. Fig. 4.

DEMONSTRATION.

Firſt, let there be drawn through the extremities B and M of the diameter AB and chord AM, the chord BLM, terminated by the tangent of the arch AM at the point T; then let there be drawn through the centre C the right lines CK and CL reſpectively parallel to the right lines BM and AM; and it is evident that AT will be twice the *tangent*, and CD or ML or LB the *coſine*, of half the arch AM.

This being premiſed, the ſimilar triangles BLC, BPM, will give, BC : BL : : BM : BP : : 2BL : BP; wherefore, $BP = \frac{2\overline{LB}^2}{BC}$; that is to ſay, $1+\textit{coſ.}\,A = 2\,\textit{coſ.}^2\,\frac{1}{2}A$. In like manner, on account of the ſimilar triangles CLB and APM, we ſhall have, CB : CL : : AM : AP; and therefore $AP = \frac{2\overline{AD}^2}{CB}$, (becauſe AM = 2AD or 2DM or 2CL); that is, $1-\textit{coſ.}\,A = 2\,\textit{ſin.}^2\,\frac{1}{2}A$. Q. E. 1°. D.

2. From the ſimilar triangles APM, BAT, and BPM, we ſhall get theſe two proportions; BP : PM : : BA : AT; or, $1+\textit{coſ.}\,A : \textit{ſin.}\,A : : 2R : 2\,\textit{tang}\,\frac{1}{2}A$; and PM : AP : : AB : AT; or, $\textit{ſin.}\,A : 1-\textit{coſ.}\,A : : 2R : 2\,\textit{tang.}\,\frac{1}{2}A$: therefore, $1+\textit{coſ.}A = \frac{\textit{ſin.}\,A\times R}{\textit{tang.}\,\frac{1}{2}A}$; and, $1-\textit{coſ.}\,A = \frac{\textit{ſin.}\,A\times\textit{tang.}\,\frac{1}{2}A}{R}$. Q. E. 2°. D.

COROLLARY.

22. It follows from hence, that $\frac{1-\text{coſ.}\,A}{1+\text{coſ.}\,A} = \frac{2\,\text{ſin.}^2\,\frac{1}{2}A}{2\,\text{coſ.}^2\,\frac{1}{2}A} = \frac{\text{tang.}^2\,\frac{1}{2}A}{RR}$; and, $\frac{1+\text{coſ.}\,A}{1-\text{coſ.}\,A} = \frac{\text{cot.}^2\,\frac{1}{2}A}{RR}$; and it likewiſe follows, by putting V and *v* for the *verſed-ſines* AP and BP of the arch AM, that $V=\frac{2\,\text{ſin.}^2\,\frac{1}{2}A}{R}$; and, $v=\frac{2\,\text{coſ.}^2\,\frac{1}{2}A}{R}$; or, $\frac{VR}{2}=\text{ſin}^2\,\frac{1}{2}A$; and, $\frac{vR}{2}=\text{coſ.}^2\,\frac{1}{2}A$.—Moreover, if the arch MG, the complement of AM, be called A, we ſhall get from the ſimilar triangles AMP, MPB, and AMB, the following formulæ*; $R+\text{ſin.}\,A = 2\,\text{ſin.}^2\,\overline{45^\circ+\frac{1}{2}A}$; $R-\text{ſin.}A=2\,\text{ſin.}^2\,\overline{45^\circ.-\frac{1}{2}A}$, and, $\frac{R+\text{ſin.}\,A}{R-\text{ſin.}\,A}=\frac{\text{ſin.}^2\,\overline{45^\circ+\frac{1}{2}A}}{\text{ſin.}^2\,\overline{45^\circ-\frac{1}{2}A}}$.

PROBLEM I.

Fig. 5. 23. *Any two arcs,* AM *and* AN, *being given; to find the ſine of their ſum and difference.*

SOLUTION.

Let the greater arc be denoted by A, and the leſs by B: let the *ſine* MP of the greater arc AM be produced till it meet the radius CN, which paſſes through the extremity of the leſs arc, in the point R; and perpendicular to the ſame radius let the right line ML, (which will be the *ſine* of the ſum of the arcs AM and AN), be ſuppoſed to be drawn. Then the ſimilar triangles CQN, CPR, will give, $CQ : QN :: CP : PR$; or, $\text{coſ.}\,B : \text{ſin.}\,B :: \text{coſ.}\,A : PR = \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B}$; therefore $RM = \text{ſin.}\,A + \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B}$: moreover, the ſimilar triangles NQC, RLM, will give, $NC : QC :: MR : ML$; or, $R : \text{coſ.}\,B :: \text{ſin.}\,A + \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B} : \text{ſin.}\,\overline{A+B} = \frac{\text{ſin.}\,A \times \text{coſ.}\,B + \text{coſ.}\,A \times \text{ſin.}\,B}{R}$. Q. E. 1°. I.

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* The proportions, from whence theſe formulæ are derived, are, AB : BM :: BM : PB; and, AB : AM :: AM : AP; or, $2R : 2\,\text{coſ.}\,\overline{45^\circ-\frac{1}{2}A} :: 2\,\text{coſ.}\,\overline{45^\circ-\frac{1}{2}A} : R+\text{ſin.}A$; and, $2R : 2\,\text{ſin.}\,\overline{45^\circ-\frac{1}{2}A} :: 2\,\text{ſin.}\,\overline{45^\circ-\frac{1}{2}A} : R-\text{ſin.}\,A$. But theſe formulæ may be otherwiſe obtained by a ſimple change of the expreſſions, $1+\text{coſ.}\,A=2\,\text{coſ.}^2\,\frac{1}{2}A$, and, $1-\text{coſ.}\,A=2\,\text{ſin.}^2\,\frac{1}{2}A$; into, $1+\text{ſin.}\,A=2\,\text{coſ.}^2\,\overline{\frac{90^\circ-A}{2}}=2\,\text{ſin.}^2\,\overline{45^\circ+\frac{1}{2}A}$, and, $1-\text{ſin.}\,A=2\,\text{ſin.}^2\,\overline{45^\circ-\frac{1}{2}A}$.

2. In order to find the *ſine* of the difference of two arcs; we ſhall now regard MN as the greater, which we have called A, and the arc AN as the leſs, which we ſhall ſtill denote by B. This being ſuppoſed, the ſimilar triangles CQN, CLO, give, $CQ : QN :: CL : LO$; or, $\text{coſ.}\,B : \text{ſin.}\,B :: \text{coſ.}\,A : LO = \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B}$; therefore $OM = \text{ſin.}\,A - \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B}$: moreover, on account of the ſimilar triangles MPO, CQN, we ſhall have, $CN : CQ :: OM : PM$; or, $R : \text{coſ.}\,B :: \text{ſin.}\,A - \frac{\text{coſ.}\,A \times \text{ſin.}\,B}{\text{coſ.}\,B} : \text{ſin.}\,\overline{A-B} = \frac{\text{ſin.}\,A \times \text{coſ.}\,B - \text{coſ.}\,A \times \text{ſin.}\,B}{R}$. Q.E.2°.I.

PROBLEM II.

24. *To find the coſine of the ſum and difference of any two arcs,* A *and* B. Fig. 5.

SOLUTION.

The triangles CQN, MPO, being ſimilar, (ſince the ſides of the one are perpendicular to thoſe of the other), we ſhall have, CQ : QN :: MP : PO; or, $\text{coſ. } B : \text{ſin. } B :: \text{ſin. } A : PO = \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B}$; therefore $CO = \text{coſ. } A - \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B}$: but on account of the ſimilar triangles CQN, CLO, we likewiſe have, CN : CQ :: CO : CL; or, $R : \text{coſ. } B :: \text{coſ. } A - \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B} : \text{coſ. } \overline{A+B} = \frac{\text{coſ. } A \times \text{coſ. } B - \text{ſin. } A \times \text{ſin. } B}{R}$. Q. E. 1°. I.

2. If we now regard the arc MN as A, and the arc AN as B; it is evident that CP will be the *coſine* of their difference. This being granted, the ſimilar triangles CQN, MLR, will give, CQ : QN :: ML : LR; or, $\text{coſ. } B : \text{ſin. } B :: \text{ſin. } A : LR = \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B}$; therefore CR, or $CL + LR = \text{coſ. } A + \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B}$: moreover, on account of the ſimilar triangles CQN, CPR, we ſhall have, CN : CQ :: RC : CP; or, $R : \text{coſ. } B :: \text{coſ. } A + \frac{\text{ſin. } A \times \text{ſin. } B}{\text{coſ. } B} : \text{coſ. } \overline{A-B} = \frac{\text{coſ. } A \times \text{coſ. } B + \text{ſin. } A \times \text{ſin. } B}{R}$. Q. E. 2°. I.

COROLLARY I.

25. From the formulæ which we have inveſtigated in the two laſt Problems, the following proportions will manifeſtly ariſe;

$\text{Sin. } \overline{A+B} : \text{ſin. } \overline{A-B} :: \text{ſin. } A \times \text{coſ. } B + \text{coſ. } A \times \text{ſin. } B : \text{ſin. } A \times \text{coſ. } B - \text{coſ. } A \times \text{ſin. } B$, and $\text{Coſ. } \overline{A+B} : \text{coſ. } \overline{A-B} :: \text{coſ. } A \times \text{coſ. } B - \text{ſin. } A \times \text{ſin. } B : \text{coſ. } A \times \text{coſ. } B + \text{ſin. } A \times \text{ſin. } B$; and if the two laſt terms in the firſt proportion be divided by $\text{coſ. } A \times \text{coſ. } B$, and the two laſt in the ſecond proportion by $\text{coſ. } A \times \text{ſin. } B$; we ſhall get, (after ſubſtituting for the quotients their equal values in the *tangents* and *cotangents* given in *art*, 11,) theſe two analogies;

Sin. $\overline{A+B}$: *ſin.* $\overline{A-B}$ :: *tang.* A + *tang.* B : *tang.* A — *tang.* B, and *Coſ.* $\overline{A+B}$: *coſ.* $\overline{A-B}$ :: *cot.* B — *tang.* A : *cot.* B + *tang.* A :: *cot.* A — *tang.* B : *cot.* A + *tang.* B. ———It is ſcarcely neceſſary to obſerve, that theſe proportions might be written under the form of an equation.

COROLLARY II.

26. Since we have, *coſ.* $\overline{A+B}$: *coſ.* $\overline{A-B}$ :: *coſ.* A × *coſ.* B — *ſin.* A × *ſin.* B : *coſ.* A × *coſ.* B + *ſin.* A × *ſin.* B‡; it follows, that we ſhall alſo have by one *ſubtraction*, *coſ.* $\overline{A-B}$ — *coſ.* $\overline{A+B}$ = 2 *ſin.* A × *ſin.* B; and by one *addition*, *coſ.* $\overline{A+B}$ + *coſ.* $\overline{A-B}$ = 2 *coſ.* A × *coſ.* B.

In like manner from the proportion, *ſin.* $\overline{A+B}$: *ſin.* $\overline{A-B}$ :: *ſin.* A × *coſ.* B + *coſ.* A × *ſin.* B : *ſin.* A × *coſ.* B — *coſ.* A × *ſin.* B‡, we ſhall find by one *addition* and *ſubtraction*, that, *ſin.* $\overline{A+B}$ + *ſin.* $\overline{A-B}$ = 2 *ſin.* A × *coſ.* B; and, *ſin.* $\overline{A+B}$ — *ſin.* $\overline{A-B}$ = 2 *coſ.* A × *ſin.* B; and therefore, if we collect theſe ſeveral expreſſions and divide them by 2, we ſhall get

Sin. A × *ſin.* B = $\frac{\textit{coſ.}\,\overline{A-B}\,-\,\textit{coſ.}\,\overline{A+B}}{2}$; and, *ſin.* A × *coſ.* B = $\frac{\textit{ſin.}\,\overline{A+B}\,+\,\textit{ſin.}\,\overline{A-B}}{2}$: *Coſ.* A × *coſ.* B = $\frac{\textit{coſ.}\,\overline{A+B}\,+\,\textit{coſ.}\,\overline{A-B}}{2}$; and, *coſ.* A × *ſin.* B = $\frac{\textit{ſin.}\,\overline{A+B}\,-\,\textit{ſin.}\,\overline{A-B}}{2}$.

COROL-

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‡ Should any difficulty ariſe with reſpect to the equations deduced from theſe analogies, it will immediately vaniſh upon conſidering, that as both the antecedents and conſequents in each are equal, the ſum or difference of the 1ſt and 2d terms muſt neceſſarily be equal to the ſum or difference of the 3d and 4th.

COROLLARY III.

27. Hence it again follows, that if we know the *ſines* and *coſines* of all the arcs below 30°, we ſhall be able to get the *ſines* and *coſines* of all the arcs above 30° to 60° by ſubtraction only; and therefore, if by this method we compute the *ſines* and *coſines* to 45°, we ſhall have all the *ſines* and *coſines* to 90°; ſince the *coſine* of an arc below 45° is the *ſine* of an arc as much above 45°. But that the truth of this Corollary may be clearly apprehended, it muſt be obſerved, that the *ſine* of 30° being half the chord of 60° will be equal to $\frac{R}{2}$; and therefore if we make $A = 30°$, $\textit{ſin.}\,A \times \textit{ſin.}\,B$ will be equal to $\frac{R \times \textit{ſin.}\,B}{2} = \frac{\textit{coſ.}\,\overline{A-B} - \textit{coſ.}\,\overline{A+B}}{2}$; and conſequently, $R \times \textit{ſin.}\,B = \textit{coſ.}\,\overline{30° - B} - \textit{coſ.}\,\overline{30° + B}$; or, $\textit{coſ.}\,\overline{30° + B} = \textit{coſ.}\,\overline{30° - B} - \textit{ſin.}\,B$: from whence $\textit{coſ.}\,\overline{30° + B}$ will become known, ſince B is known by the hypotheſis, and leſs than 30°. In like manner, ſince $\textit{ſin.}\,A \times \textit{coſ.}\,B = \frac{\textit{ſin.}\,\overline{A+B} + \textit{ſin.}\,\overline{A-B}}{2}$, we ſhall find, $\textit{coſ.}\,B = \textit{ſin.}\,\overline{30° + B} + \textit{ſin.}\,\overline{30° - B}$; and, $\textit{ſin.}\,\overline{30° + B} = \textit{coſ.}\,B - \textit{ſin.}\,\overline{30° - B}$: therefore it is manifeſt that the Tables of *ſines* and *coſines* may be eaſily calculated from this Corollary.

COROLLARY IV.

28. If we ſuppoſe the arc B ſucceſſively equal to A, 2 A, 3 A, &c. the *ſine* of the arc A to be always denoted by *s*, and its *coſine* by *c*, and the expreſſions of the *ſines* of multiple arcs in *ſines* and *coſines* to be repreſented by *ſin.* A, *ſin.* 2 A, *ſin.* 3 A, *ſin.* 4 A———*ſin.* *n* A, it will be eaſy to conſtruct

the following Tables by means of the formula in *Prob.* 1. *Sin.* $\overline{A+B}=$ *ſin.* $A\times$ *coſ.* $B+$ *coſ.* $A\times$ *ſin.* B.

Sin. $A=\sqrt{rr-cc}$.
Sin. $2A=2c\sqrt{rr-cc}$.
Sin. $3A=\overline{4cc-1}\times\sqrt{rr-cc}$.
Sin. $4A=\overline{8c^3-4c}\times\sqrt{rr-cc}$.
Sin. $5A=\overline{16c^4-12c^2+1}\times\sqrt{rr-cc}$.
Sin. $6A=\overline{32c^5-32c^3+6c}\times\sqrt{rr-cc}$.
Sin. $7A=\overline{64c^6-80c^4+24c^2-1}\times\sqrt{rr-cc}$.
Sin. $8A=\overline{128c^7-192c^5+80c^3-10c}\times\sqrt{rr-cc}$.
&c.

OR,

† *Sin.* $A=s$.
Sin. $2A=2s\sqrt{rr-ss}$.
Sin. $3A=3s-4s^3$.
Sin. $4A=\overline{4s-8s^3}\times\sqrt{rr-ss}$.
Sin. $5A=5s-20s^3+16s^5$.
Sin. $6A=6s-32s^3+32s^5\times\sqrt{rr-ss}$.
Sin. $7A=7s-56s^3+112s^5-64s^7$.
Sin. $8A=\overline{8s-80s^3+192s^5-128s^7}\times\sqrt{rr-ss}$.
Sin. $9A=9s-120s^3+432s^5-576s^7+256s^9$.
&c.

It is eaſy to perceive that all the terms of theſe equations might be rendered homogeneous, by ſubſtituting therein the different powers of the radius or *ſine* total, which we have not thought neceſſary to be ſpecified. It is likewiſe evident that the firſt Table expreſſes the *ſines* of multiple arcs in *coſines*, and the ſecond in *ſines*, of the ſimple arc. —But that theſe equations may have all the generality which 'tis poſſible to give them, we ſhall ſubjoin the following general formulæ for the two Tables; obſerving, that the ſecond Table muſt have two general formulæ to expreſs it, ſince $\sqrt{rr-ss}$ is found in all the terms of the even, but not in thoſe of the odd, ranks. As to the method of finding theſe formulæ, it is deduced from the conſideration of the coefficients and exponents of the ſeveral terms.

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† See *Maſeres*'s Trigonometry.

A. *General formula for the firſt Table*, n *being any number whatever.*

$$\text{Sin. } n\,\text{A} = \Big(2^{n-1}c^{n-1} - \overline{n-2}\times 2^{n-3}c^{n-3} + \frac{\overline{n-3}\times\overline{n-4}}{1.2}\,2^{n-5}c^{n-5} - \frac{\overline{n-4}\times\overline{n-5}\times\overline{n-6}}{1.2.3}\times 2^{n-7}c^{n-7} + \frac{\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times\overline{n-8}}{1.2.3.4}\,2^{n-9}c^{n-9} - \&\text{c.}\Big)\times\sqrt{rr-cc}.$$

B. *Firſt general formula for the ſecond Table*, n *being any odd number.*

$$\text{Sin. } n\text{A} = ns - \frac{n\times\overline{nn-1}}{1.2.3}\,s^3 + \frac{n\times\overline{nn-1}\times\overline{nn-9}}{1.2.3.4.5}\,s^5 - \frac{n\times\overline{nn-1}\times\overline{nn-9}\times\overline{nn-25}}{1.2.3.4.5.6.7}\,s^7 + \&\text{c.}$$

C. *Second general formula for the ſame Table*, n *being any even number.*

$$\text{Sin. } n\text{A} = \Big(ns - \frac{n\times\overline{nn-4}}{1.2.3}\,s^3 + \frac{n\times\overline{nn-4}\times\overline{nn-16}}{1.2.3.4.5}\,s^5 - \frac{n\times\overline{nn-4}\times\overline{nn-16}\times\overline{nn-36}}{1.2.3.4.5.6.7}\,s^7 + \ldots\ldots + 2^{n-1}s^{n-1}\Big)\times\sqrt{rr-ss}.$$

COROL-

COROLLARY V.

29. Moreover, if we ſtill ſuppoſe the arc B ſucceſſively equal to A, 2A, 3A, 4A, &c. &c. &c. and proceed in a ſimilar manner with the formula of the *coſine* of the ſum of two arcs, found in *Prob.* II. we ſhall again form the two following Tables; one of which expreſſes the *coſines* of multiple arcs in *coſines*, and the other in *ſines* of the ſimple arc.

Coſ. $A=c$.
Coſ. $2A=2cc-1$.
Coſ. $3A=4c^3-3c$.
Coſ. $4A=8c^4-8c^2+1$.
Coſ. $5A=16c^5-20c^3+5c$.
Coſ. $6A=32c^6-48c^4+18c^2-1$.
Coſ. $7A=64c^7-112c^5+56c^3-7c$.
&c.

OR,

Coſ. $A=\sqrt{rr-ss}$.
Coſ. $2A=-2ss+rr$.
Coſ. $3A=\overline{-4ss+rr}\times\sqrt{rr-ss}$.
Coſ. $4A=8s^4-8s^2+1$.
Coſ. $5A=\overline{16s^4-12s^2+1}\times\sqrt{rr-ss}$.
Coſ. $6A=-32s^6+48s^4-18s^2+1$.
Coſ. $7A=\overline{-64s^6+80s^4-24s^2+1}\times\sqrt{rr-ss}$.
&c.

D. *General formula for the firſt Table,* n *being any number whatever.*

$$r^{n-1}\text{ Coſ. }nA=2^{n-1}c^n-nr^2\,2^{n-3}c^{n-2}+\frac{n\times\overline{n-3}}{1.2}r^4\,2^{n-5}c^{n-4}-\frac{n\times\overline{n-4}\times\overline{n-5}}{1.2.3}r^6\,2^{n-7}c^{n-6}+\frac{n\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}}{1.2.3.4}r^8\,2^{n-9}c^{n-8}-\&c.$$

If the arc denoted by A be obtuſe, its *coſine* becomes negative, and then all the formulæ wherein the letter *c* is involved to odd powers, change their ſigns; as will likewiſe the terms of the general formula, when *n* is an odd number.

F. *General formula for the ſecond Table*, n *being any odd number*.

$$\text{Coſ. } nA = \mp\left(\mp 2^{n-1}s^{n-1} \pm \overline{n-2} \times 2^{n-3}s^{n-3} \pm \frac{\overline{n-3} \times \overline{n-4}}{1.2} 2^{n-5}s^{n-5} \mp \text{\&c.}\right) \times \sqrt{rr-ss}.$$

G. *General formula for the ſame Table, when* n *is any even number.*

$$r^{n-1} \text{ Coſ. } nA = \pm 2^{n-1}s^{n} \mp nr^2 2^{n-3}s^{n-2} \pm \frac{n \times \overline{n-3}}{1.2} r^4 2^{n-5}s^{n-4} \mp \frac{n \times \overline{n-4} \times \overline{n-5}}{1.2.3} r^6 2^{n-7}s^{n-6} \pm \frac{n \times \overline{n-5} \times \overline{n-6} \times \overline{n-7}}{1.2.3.4} r^8 2^{n-9}s^{n-8} \pm \text{\&c.}$$

The higher ſigns take place in the firſt formula, when *n* is one of the odd numbers 1, 5, 9, 13, &c. and the lower ſigns, when it is one of the numbers 3, 7, 11, &c.

In like manner in the ſecond formula the higher ſigns prevail, when the numbers are evenly even; and the lower ſigns when they are evenly odd.

SCHOLIUM.

30. We might now proceed to deduce from the foregoing general formulæ, ſeveral uſeful and important truths concerning the nature of the roots of equations; but fearing by this means to extend our Work to too great a length, we ſhall refer our readers to the works of Mr. *Euler*, where this matter is fully and ſatisfactorily handled; and only apply

apply them to the theory of powers of the *ſines* and *coſines*, in two Problems: the firſt of which requires, when "*Any power of a ſine or coſine is given, to find its expreſſion in ſines and coſines of the ſimple arc and its multiples*;" and the ſecond (the bare ſpecification whereof will be ſufficient), "*To expreſs the ſine or coſine of any multiple arc by powers of the ſine or coſine of the ſimple arc.*" The formulæ, which Mr. *Euler* hath given upon this theory, he deduced from thoſe in *art.* 26; but to us it appears, that they are much more eaſily deducible from the preceding formulæ.

PROBLEM III.

31. *To expreſs powers of the ſine and coſine of any arc by ſines and coſines of this arc and its multiples.*

SOLUTION.

This Problem conſiſts of two parts, one for the *ſines*, and the other for the *coſines*; but the formulæ, which we ſhall firſt inveſtigate, ſhall be thoſe which may contain the values of the powers of the *ſines*; and this we ſhall do by means of the two laſt Tables in *Corol.* IV. and V. wherein the letter *s* was ſuppoſed to denote the *ſine* of the ſimple arc. Now it will appear with a very little conſideration, that the terms of the odd ranks of the ſecond Table in *art.* 28 will give the values of the odd powers of the letter *s*, and the terms of the even ranks in the ſecond Table of *Corollary* V. thoſe of its even powers; after exterminating ſuch powers as may be found in the laſt terms.——Thus from $\text{ſin.}\,A=s$, we get, $\text{ſin.}\,A=\text{ſin.}\,A$: alſo from the equation, $\text{coſ.}\,2A=-2s^2+rr$, we get, $2\,\text{ſin.}^2\,A=rr-\text{coſ.}\,2A$: again, from the equation, $\text{ſin.}\,3A=3s-4s^3$, in the ſecond Table of *Corol.* IV.

deduce, $4\,\text{ſin.}^3 A = 3\,\text{ſin.}\,A - \text{ſin.}\,3A$; likewiſe from, $\text{coſ.}\,4A = 8\,\text{ſin.}^4 - 8\,\text{ſin.}^2 + 1$, in the ſecond Table of *Corol.* V. we ſhall get, $8\,\text{ſin.}^4 A = \text{coſ.}\,4A + 8\,\text{ſin.}^2 A - 1 = \text{coſ.}\,4A - 4\,\text{coſ.}\,2A + 3$, by ſubſtituting for $8\,s.^2 A$ its value, $4rr - 4\,\text{coſ.}\,2A$, as obtained from the above equation, $2\,\text{ſin.}^2 A = rr - \text{coſ.}\,2A$: and by purſuing this proceſs, the following Table will be eaſily formed.

$\text{Sin.}\,A = \text{ſin.}\,A.$
$2\,\text{Sin.}^2 A = r^2 - \text{coſ.}\,2A.$
$4\,\text{Sin.}^3 A = 3\,\text{ſin.}\,A - \text{ſin.}\,3A.$
$8\,\text{Sin.}^4 A = 3 - 4\,\text{coſ.}\,2A + \text{coſ.}\,4A.$
$16\,\text{Sin.}^5 A = 10\,\text{ſin.}\,A - 5\,\text{ſin.}\,3A + \text{ſin.}\,5A.$
$32\,\text{Sin.}^6 A = 10 - 15\,\text{coſ.}\,2A + 6\,\text{coſ.}\,4A - \text{coſ.}\,6A.$
$64\,\text{Sin.}^7 A = 35\,\text{ſin.}\,A - 21\,\text{ſin.}\,3A + 7\,\text{ſin.}\,5A - \text{ſin.}\,7A.$
$128\,\text{Sin.}^8 A = 35 - 56\,\text{coſ.}\,2A + 28\,\text{coſ.}\,4A - 8\,\text{coſ.}\,6A + \text{coſ.}\,8A.$
$256\,\text{Sin.}^9 A = 126\,\text{ſin.}\,A - 84\,\text{ſin.}\,3A + 36\,\text{ſin.}\,5A - 9\,\text{ſin.}\,7A + \text{ſin.}\,9A.$
&c.

In the above Table it is manifeſt, that the odd powers of the *ſine* of the arc A are all expreſſed in *ſines*, but its even powers in *coſines*, of its multiple arcs. The law of the coefficients is likewiſe very obſervable, being evidently the ſame with that of the coefficients of a binomial raiſed to each of the powers; except that in the even powers the abſtract number, which is not multiplied by a *coſine* of A, is only half the coefficient of the correſponding term in the like power of the binomial. Hence it will appear, that in order to expreſs this Table of powers, we muſt neceſſarily have two general formulæ.

H. *Firſt general formula for the powers of ſin.* A *when* n *is odd, beginning at the laſt Terms of the Table.*

$$2^{n-1}\,\text{Sin.}^n A = \pm\,\text{ſin.}\,nA \pm n\,\text{ſin.}\,\overline{n-2}\,A \pm \frac{n \times \overline{n-1}}{1.\,2}\,\text{ſin.}\,\overline{n-4}\,A \mp \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\,2.\,3} \times \text{ſin.}\,\overline{n-6}\,A \pm \&c. \ldots \ldots \ldots \pm \frac{n \times \overline{n-1} \ldots \times \overline{n-2}\,\&c.}{1.\,2.\,3\,\&c.}\,\text{ſin.}\,A.$$

K. *Second general formula for the powers of the ſame ſine, when* n *is any even number.*

$$2^{n-1}\ \text{Sin.}^{n} A = \pm\ \text{coſ.}\ n A \mp n\ \text{coſ.}\ \overline{n-2}\, A \pm \frac{n \times \overline{n-1}}{1.\ 2}\ \text{coſ.}\ \overline{n-4}\, A \mp \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\ 2.\ 3}\ \text{coſ.}\ \overline{n-6}\, A \ldots\ldots\ldots \mp \tfrac{1}{2} \times \frac{n \times \overline{n-1} \times \overline{n-2}\ \&c.}{1.\ 2.\ 3\ \&c.}\ \text{coſ.}\ 0 A.$$

Each of the two preceding formulæ hath two ſigns; and of theſe the upper muſt be uſed, when n is an odd number and equal to $4m+1$; but the lower, when it is equal to $4m-1$; m repreſenting any number whatever.——In like manner, the higher ſigns muſt be uſed in the ſecond formula, when n is equal to $4m$, m denoting any number at pleaſure; but the lower ſigns, when n is equal to $2m$, and m any odd number.

32. Now in the ſecond place, to find a ſimilar formula for the ſucceſſive powers of the *coſine* of any arc, we muſt apply the firſt Table of *Corol.* V. exactly in the ſame manner as we before applied the two laſt Tables of *Corol.* IV. and V. and we ſhall by that means eaſily form the ſubſequent Table;

$\text{Coſ.}\ A = \text{coſ.}\ A.$

$2\ \text{Coſ.}^{2} A = 1 + \text{coſ.}\ 2 A.$

$4\ \text{Coſ.}^{3} A = 3\ \text{coſ.}\ A + \text{coſ.}\ 3 A.$

$8\ \text{Coſ.}^{4} A = 3 + 4\ \text{coſ.}\ 2 A + \text{coſ.}\ 4 A.$

$16\ \text{Coſ.}^{5} A = 10\ \text{coſ.}\ A + 5\ \text{coſ.}\ 3 A + \text{coſ.}\ 5 A.$

$32\ \text{Coſ.}^{6} A = 10 + 15\ \text{coſ.}\ 2 A + 6\ \text{coſ.}\ 4 A + \text{coſ.}\ 6 A.$

$64\ \text{Coſ.}^{7} A = 35\ \text{coſ.}\ A + 21\ \text{coſ.}\ 3 A + 7\ \text{coſ.}\ 5 A + \text{coſ.}\ 7 A.$

&c.

Then if we regard the laſt terms of theſe equations as the firſt; or which is the ſame, take them all backwards, we ſhall readily obtain the following general formula;

L. $2^{n-1}\,\text{Coſ}^{n}\,A = \text{coſ.}\,nA + n\,\text{coſ.}\,\overline{n-2}\,A + \frac{n\times\overline{n-1}}{1.\,2}\,\text{coſ.}\,\overline{n-4}\,A + \frac{n\times\overline{n-1}\times\overline{n-2}}{1.\,2.\,3}\,\text{coſ.}\,\overline{n-6}\,A \ldots \ldots\ldots + \frac{n\times\overline{n-1}\times\overline{n-2}\ \&c.}{1.\,2.\,3}\,\text{coſ.}\,\overline{n-n}\,A$, or $\text{coſ.}\,A$; according as n is an even or odd number.

SCHOLIUM.

33. Though we have already ſolved the converſe of the laſt Problem; yet as it will admit of other different ſolutions, by taking the equations of the Table in *art.* 28 and 29. backwards, and ſeeking the laws of their terms, we ſhall likewiſe give theſe ſolutions.

PROBLEM IV.

34. *To find general formulæ for transforming the ſine or coſine of any multiple of an arc* A, *into powers of the ſine or coſine of the ſimple arc.*

SOLUTION.

This Problem conſiſts of ſeveral parts; for if the *ſine* of a multiple arc be given, it is manifeſt that it may be expreſſed in *ſines* or *coſines*, according as we make uſe of the ſecond or firſt Table in *art.* 28: it is likewiſe evident, that in both theſe caſes different formulæ will be requiſite, according as we begin the ſeries by the firſt or laſt terms; and that in the third place, when we begin the ſeries by the laſt terms, the general formulæ will again be different, according as the number n is even or odd. Now if we ſuppoſe the ſeries to begin by the firſt terms, and it be required to expreſs multiple *ſines* in powers of the *coſine* of the ſimple arc, the formula A in *art.* 28 will ſolve the Problem, whether n repreſents an even or odd number; whilſt the formulæ

B and C in the ſame article will give the general values of multiple *ſines* in powers of the *ſine* of the ſimple arc, viz. B when n is any odd number, and C when it is an even one: but the general formulæ for expreſſing multiple *ſines* in powers of the *ſine* of the ſimple arc, when the ſeries are ſuppoſed to begin by the laſt terms, remain ſtill to be obtained; and theſe will manifeſtly be two, one for n when even, and the other for it when odd.

M. *General formula for reducing any multiple ſine into powers of the ſine of the ſimple arc,* n *being an even number.*

$$\text{Sin. } n\,A = (\mp 2^{n-1} s^{n-1} \pm \overline{n-2} \times 2^{n-3} s^{n-3} \mp \frac{\overline{n-3} \times \overline{n-4}}{1.\,2} 2^{n-5} s^{n-5} \pm \frac{\overline{n-4} \times \overline{n-5} \times \overline{n-6}}{1.\,2.\,3} \times 2^{n-7} s^{n-7} \mp \&c. \ldots\ldots \pm ns) \sqrt{rr-ss}.$$

N. *General formula for reducing a multiple ſine into powers of the ſine of the ſimple arc,* n *being any odd number.*

$$\text{Sin. } n\,A = \mp 2^{n-1} s^{n} \pm n \times 2^{n-3} s^{n-2} \mp \frac{n \times \overline{n-3}}{1.\,2} 2^{n-5} s^{n-4} \pm \frac{n \times \overline{n-4} \times \overline{n-5}}{1.\,2.\,3} 2^{n-7} s^{n-6} \mp \frac{n \times \overline{n-5} \times \overline{n-6} \times \overline{n-7}}{1.\,2.\,3.\,4} 2^{n-9} s^{n-8} \pm \frac{n \times \overline{n-6} \times \overline{n-7} \times \overline{n-8} \times \overline{n-9}}{1.\,2.\,3.\,4.\,5} 2^{n-11} s^{n-10} \mp \&c.$$

O. *General formula for reducing a multiple coſine into powers of the coſine of the ſimple arc,* n *being any even number.*

Coſ.

$$\text{Coſ.}\ n\,\mathrm{A} = \pm r^{n} \mp \frac{n^2}{1.2} r^{n-2}\,\text{coſ.}^2\,\mathrm{A} \pm \frac{n^2 \times \overline{n^2-4}}{1.2.3.4} r^{n-4}\,\text{coſ.}^4\,\mathrm{A} \mp \frac{n^2 \times \overline{n^2-4} \times \overline{n^2-16}}{1.2.3.4.5.6} r^{n-6}\,\text{coſ.}^6\,\mathrm{A} \pm \frac{n^2 \times \overline{n^2-4} \times \overline{n^2-16} \times \overline{n^2-36}}{1.2.3.4.5.6.7.8} r^{n-8}\,\text{coſ.}^8\,\mathrm{A} \mp \text{\&c.}$$

P. *General formula fcr* n, *when any odd number.*

$$\text{Coſ.}\ n\,\mathrm{A} = \pm n r^{n-1}\,\text{coſ.}\,\mathrm{A} \mp \frac{n \times \overline{n^2-1}}{1.2.3} r^{n-3}\,\text{coſ.}^3\,\mathrm{A} \pm \frac{n \times \overline{n^2-1} \times \overline{n^2-9}}{1.2.3.4.5} r^{n-5}\,\text{coſ.}^5\,\mathrm{A} \mp \frac{n \times \overline{n^2-1} \times \overline{n^2-9} \times \overline{n^2-25}}{1.2.3.4.5.6.7} r^{n-7}\,\text{coſ.}^7\,\mathrm{A} \pm \frac{n \times \overline{n^2-1} \times \overline{n^2-9} \times \overline{n^2-25} \times \overline{n^2-49}}{1.2.3.4.5.6.7.8.9} r^{n-9}\,\text{coſ.}^9\,\mathrm{A} \mp \text{\&c.}$$

Theſe two laſt formulæ are eaſily deduced from the firſt Table of *Corol.* V. (*art.* 29), by obſerving the law of the coefficients for the terms of the even and odd ranks.—In the firſt formula M of the *ſines*, the higher ſigns muſt be taken when n is any of the even numbers denoted by $4\,m$, and m any number at pleaſure; but the lower ſigns, when it is any of the numbers repreſented by $2\,m$, and m any odd number.—In like manner, in the ſecond formula N, the higher ſigns muſt be uſed for all the odd numbers denoted by $4\,m-1$; and the lower for thoſe repreſented by $4\,m+1$; m being any number whatever. We may moreover eaſily perceive, that in the third ſeries O the *coſine* will be poſitive or negative, according as n is equal to $4\,m$ or $2\,m$; m being any number at pleaſure in the firſt caſe, and any odd one in the ſecond; and that laſtly, when n is odd, the *coſine* will be poſitive or negative, according as n is equal to $4\,m+1$, or $4\,m-1$.

Of

Of the uſe of imaginary Factors in the theory of Sines and Coſines.

35. All the ſeries, which we have hitherto given, were obtained by immediate deduction from the two Problems concerning the *ſine* and *coſine* of the ſum and difference of any two arcs, by ſuppoſing theſe arcs to be equal: but there are ſeveral other methods ſtill remaining whereby ſeries of a different nature from the foregoing may be diſcovered; of which we ſhall however only ſubjoin that wherein *imaginary Factors* are concerned; not becauſe we deem expreſſions abſurd in themſelves to be preferable to others, but becauſe they may ſometimes prove peculiarly ſerviceable in the abridgement of calculations, and the diſcovery of important truths.

PROBLEM V.

36. *To find the Factors of the ſum of two ſquares, as* $aa+bb$; *or, which is the ſame, to diſcover how* a *and* b *muſt be combined by multiplication, ſo that their product may be,* $aa+bb$.

SOLUTION.

Let A and B be the indeterminate quantities, which ought to affect the roots, a and b, of the ſquares, ſo that their product may be $aa+bb$, and let us ſuppoſe the Factors to be, $a+Ab$ and $aB+b$. Then multiplying theſe two Factors together we ſhall get, $aaB+ab\,AB+ab+A\,bb=aa+bb$; and if we compare the terms of this equation together, and ſuppoſe $aa\,B=aa$; or $A\,bb=bb$, we ſhall find B and A to be each $=1$; from whence it is eaſily inferred, that if the quantities A and B are poſſible, A muſt be equal to B, and $AB=AA=BB$: but ſince in the ſum $aa+bb$ the term ab is wanting, it neceſſarily follows that, $ab+ab\,AB$; or, $ab+ab\,AA$, or, $ab+ab\,BB$, muſt be $=0$; that is to ſay, $AA+1$ or $BB+1=0$; which is impoſſible, ſince we have already ſeen that AA or $BB=1$: therefore the Factors cannot have the form ſuppoſed; and conſequently the aſſumed quantity cannot be reſolved into Factors; or, which is the ſame, is not a product of the quantities a and b in whatever manner they may be combined. However, if we would ſolve the equations, $AA+1=0$ and $BB+1=0$, notwithſtanding the abſurdity they imply, as before obſerved; we ſhall have, AA or $BB=-1$; and by extracting the roots, $A=\pm\sqrt{-1}$, and $B=\pm\sqrt{-1}$. Theſe expreſſions therefore, which Geometers have called imaginary, are not any real quantities, but only ſigns of an abſurd ſuppoſition, wherein we regard that as the product of two quantities which in reality is not: nevertheleſs as theſe ſigns may have their uſe, no juſt reaſon can be alledged why they ſhould not be employed in calculations; and if ſo, the imaginary Factors of $aa+bb$ will be, $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$; or, $b+a\sqrt{-1}$ and $b-a\sqrt{-1}$, or, $-a+b\sqrt{-1}$ and $-a-b\sqrt{-1}$. Q. E. I.

SCHOLIUM.

37. But though we have ſaid above that $aa+bb$ cannot be a product of the quantities a and b, yet it may perhaps be aſked whether it be not poſſible to find this ſum of the ſquares of a and b by ſome other combination, as, $a+b$ and $a-b$. To try this, let each of the quantities $a+b$ and $a-b$ be ſquared, and we ſhall get, $aa+2ab+bb$ and $aa-2ab+bb$; then let theſe ſquares be added together and their ſum divided by 2, and we ſhall by

that means obtain the propoſed ſum, $aa+bb$; from whence it is manifeſt that $aa+bb$ is not a product of the quantities $a+b$ and $a-b$, but only half the ſum of their ſquares.

We might extend this theory much farther: however the *little*, which we have ſaid upon it, was in our opinion abſolutely neceſſary to be given, ſince the nature of *Imaginaries* hath not been clearly treated of and explained by any Author, at leaſt by any that we know of, notwithſtanding they have been introduced into the *ſines* and *coſines*. Beſides, there are certain peculiar relations found to prevail between curves of the hyperbolical and elliptical kind, and between logarithms and circular arcs, whereof it will be ſcarcely poſſible to form true and preciſe ideas, without calling in the aſſiſtance of what hath been here ſaid upon imaginary expreſſions.

COROLLARY I.

38. It follows from hence, that if we would find the imaginary Factors of, $\text{ſin.}^2 A+\text{coſ.}^2 A=RR$, we ſhall have, $\overline{\text{coſ.}\,A+\text{ſin.}\,A\sqrt{-1}}\times\overline{\text{coſ.}\,A-\text{ſin.}\,A\sqrt{-1}}=RR$. In like manner, if we ſuppoſe another arc B, and would make uſe of imaginary Factors, we ſhall find that, $\overline{\text{coſ.}\,A+\text{ſin.}\,A\sqrt{-1}}\times\overline{\text{coſ.}\,B+\text{ſin.}\,B\sqrt{-1}}=\text{coſ.}\,A\times\text{coſ.}\,B-\text{ſin.}\,A\times\text{ſin.}\,B+\sqrt{-1}\times\overline{\text{coſ.}\,A\times\text{ſin.}\,B+\text{ſin.}\,A\times\text{coſ.}\,B}=\text{coſ.}\,\overline{A+B}+\sqrt{-1}\times\text{ſin.}\,\overline{A+B}$ by *art.* 23 and 24; and that, $\overline{\text{coſ.}\,A-\text{ſin.}\,A\sqrt{-1}}\times\overline{\text{coſ.}\,B-\text{ſin.}\,B\sqrt{-1}}=\text{coſ.}\,\overline{A+B}-\sqrt{-1}\times\text{ſin.}\,\overline{A+B}$. Alſo by a ſimilar calculation we ſhall find that, $\overline{\text{coſ.}\,A\pm\sqrt{-1}\,\text{ſin.}\,A}\times\overline{\text{coſ.}\,B\pm\sqrt{-1}\,\text{ſin.}\,B}\times\overline{\text{coſ.}\,C\pm\sqrt{-1}\,\text{ſin.}\,C}=\text{coſ.}\,\overline{A+B+C}\pm\sqrt{-1}\,\text{ſin.}\,\overline{A+B+C}$.

COROLLARY II.

39. Hence it again follows, that if we ſuppoſe the arcs A and B to be equal, and take two Factors, we ſhall have, $\overline{\text{coſ. } A \pm \text{ſin. } A\sqrt{-1}}^{2} = \text{coſ. } 2A \pm \text{ſin. } 2A\sqrt{-1}$: if we take three Factors, we ſhall have, $\overline{\text{coſ. } A \pm \text{ſin. } A\sqrt{-1}}^{3} = \text{coſ. } 3A \pm \text{ſin. } 3A\sqrt{-1}$; and conſequently in general, $\overline{\text{coſ. } A \pm \text{ſin. } A\sqrt{-1}}^{n} = \text{coſ. } n A \pm \text{ſin. } n A\sqrt{-1}$.

COROLLARY III.

40. From the laſt equation we ſhall get by tranſpoſition, $\text{ſin. } n A\sqrt{-1} = \overline{\text{coſ. } A + \text{ſin. } A\sqrt{-1}}^{n} - \text{coſ. } n A$; and alſo on account of the ſign —, $\text{ſin. } n A\sqrt{-1} = \text{coſ. } n A - \overline{\text{coſ. } A - \text{ſin. } A\sqrt{-1}}^{n}$: then if we add theſe two equations together and divide by $2\sqrt{-1}$, we ſhall get the following expreſſion for the *ſine* of any multiple arc;

$$\text{Sin. } n A = \frac{\overline{\text{coſ. } A + \text{ſin. } A\sqrt{-1}}^{n} - \overline{\text{coſ. } A - \text{ſin. } A\sqrt{-1}}^{n}}{2\sqrt{-1}}$$

We might likewiſe find that

$$\text{Coſ. } n A = \frac{\overline{\text{coſ. } A + \text{ſin. } A\sqrt{-1}}^{n} + \overline{\text{coſ. } A - \text{ſin. } A\sqrt{-1}}^{n}}{2}$$

COROLLARY IV.

41. Now if we raiſe each of theſe binomials to the power *n* by the general formula, all the terms affected with *Imaginaries* will be found to deſtroy, and the two following general ſeries remain; one whereof expreſſes the *ſines* of multiple arcs, and the other the *coſines*.

Q. *Firſt formula.*

$$\textit{Sin. } n\,A = n\,\textit{coſ.}^{n-1} A \times \textit{ſin. } A - \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\,2.\,3}\,\textit{coſ.}^{n-3} A \times \textit{ſin.}^{3} A + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.\,2.\,3.\,4.\,5}\,\textit{coſ.}^{n-5} A \times \textit{ſin.}^{5} A - \&c.$$

R. *Second formula.*

$$\textit{Coſ. } n\,A = \textit{coſ.}^{n} A - \frac{n \times \overline{n-1}}{1..2}\,\textit{coſ.}^{n-2} A \times \textit{ſin.}^{2} A + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.\,2.\,3.\,4}\,\textit{coſ.}^{n-4} A \times \textit{ſin.}^{4} A - \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}}{1.\,2.\,3.\,4.\,5.\,6}\,\textit{coſ.}^{n-6} A \times \textit{ſin.}^{6} A + \&c.$$

COROLLARY V.

42. If we ſuppoſe the arc A to be indefinitely ſmall, we ſhall have, $\textit{ſin. } A = A$, and, $\textit{coſ.} A = r$ or 1: therefore that the arc $n\,A$ may become an aſſignable quantity, n itſelf muſt neceſſarily become indefinitely great, and conſequently the products, $n \times \overline{n-1}$, $n \times \overline{n-1} \times \overline{n-2}$ *&c.* be reduced to the powers, n^2, n^3, *&c.* Hence if we put the arc $n\,A = c$, we ſhall have, (ſince $\textit{ſin. } A = A$, and $A = \frac{c}{n}$), $\textit{ſin.}^{2} A = \frac{c^2}{n^2}$, $\textit{ſin.}^{3} A = \frac{c^3}{n^3}$, *&c. &c.* whilſt all the powers of *coſ.* A will remain equal to unity; by which means the formulæ in the preceding Corollary, will be reduced to the two following ones, whereby it will be eaſy to expreſs the *ſine* and *coſine* of any arc in parts of that arc, or in decimals of the radius; that is, to calculate the [illegible] *ſine* and *coſine* of any arc*.

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* See "*Expoſition du Calcul Aſtronomique,*" par Mr. DE LA LANDE, page 21.

S. *Firſt formula.*

$$Sin.\, c = c - \frac{c^3}{1.\,2.\,3} + \frac{c^5}{1.\,2.\,3.\,4.\,5} - \frac{c^7}{1.\,2.\,3.\,4.\,5.\,6.\,7} + \frac{c^9}{1.\,2.\,3.\,4.\,5.\,6.\,7.\,8.\,9} - \&c.$$

T. *Second formula.*

$$Coſ.\, C = 1 - \frac{c^2}{1.\,2} + \frac{c^4}{1.\,2.\,3.\,4} - \frac{c^6}{1.\,2.\,3.\,4.\,5.\,6} + \frac{c^8}{1.\,2.\,3.\,4.\,5.\,6.\,7.\,8} - \&c.$$

Theſe fomulæ, it is manifeſt, will give the *ſine* and *coſine* ſo much the ſooner, as they converge the faſter; that is, as the number of degrees, which the arc denoted by *c* contains, is the leſs. ——There are ſome other ſeries of a ſimilar nature with theſe, but as they are not immediately connected with the buſineſs in hand, we ſhall go on to

PROBLEM VI.

Fig. 6. 43. *To find the tangent of the ſum or difference of two given arcs* A *and* B.

SOLUTION.

Let AM and AN be the given arcs, the *tangents* of which, AG and AL, denote by t and θ; and it is evident that the line NT, drawn perpendicularly to the end of the radius CN, will be the *tangent* of the ſum of theſe arcs; which denote by T: but to find the *tangent* of the difference of two arcs A and B, let the arc MN be regarded as A, and the arc AM as B; and AL will manifeſtly be the *tangent* of their difference; which call τ. —— This being premiſed, let fall the perpendiculars, AQ, GP, upon the radius

CN: then on account of the ſimilar triangles, CRN, CAQ, we ſhall have, CR : RN :: CA : AQ; or, $\sqrt{rr+\theta\theta} : \theta :: r : \frac{r\theta}{\sqrt{rr+\theta\theta}}$: likewiſe on account of the ſimilar triangles, AQL, GPL, we ſhall have, AL : AQ :: GL : GP; or, $\theta : \frac{r\theta}{\sqrt{rr+\theta\theta}} :: t+\theta : \frac{\overline{t+\theta}\times r}{\sqrt{rr+\theta\theta}}$: moreover, ſince CL : AL :: AL : QL, we ſhall get, $QL = \frac{\theta\theta}{\sqrt{rr+\theta\theta}}$; and alſo from the ſimilar triangles, AQL, GPL, AL : QL :: GL : PL; or, $\theta : \frac{\theta\theta}{\sqrt{rr+\theta\theta}} :: t+\theta : \frac{\overline{t+\theta}\times\theta}{\sqrt{rr+\theta\theta}}$; and conſequently, CP, or CL — PL $= \sqrt{rr+\theta\theta} - \frac{\overline{t+\theta}\times\theta}{\sqrt{rr+\theta\theta}} = \frac{rr-\theta t}{\sqrt{rr+\theta\theta}}$. Now by means of theſe expreſſions it will be eaſy to find that of the *tangent* NT: for becauſe of the ſimilar triangles, GCP, CNT, we ſhall have, CP : PG :: CN : NT; or algebraically, $\frac{rr-\theta r}{\sqrt{rr+\theta\theta}} : \frac{\overline{t+\theta}\times r}{\sqrt{rr+\theta\theta}} :: r : \frac{rr\times\overline{\theta+t}}{rr-\theta t} = T = tang.\ \overline{A+B}$; whence we deduce, by making the radius equal to unity, *Tang.* $\overline{A+B} = \frac{tang.\ A + tang.\ B}{1 - tang.\ A \times tang.\ B}$. Q. E. 1°. I.

2. In order to obtain the *tangent* of the difference of the arcs A and B, let the laſt equation, $T = \frac{rr\times\overline{\theta+t}}{rr-\theta t}$, be reſumed, and one of the tangents θ or t be conſidered as unknown; and we ſhall get by the common rules of Algebra, t or $\tau = \frac{rr\times\overline{T-\theta}}{rr+\theta T}$;

that is, by making the radius equal to unity, r or $\mathit{Tang.}\ \overline{A-B} = \frac{\mathit{tang.}\,A - \mathit{tang.}\,B}{1+\mathit{tang.}\,A \times \mathit{tang.}\,B}$. Q. E. 2°. I.

SCHOLIUM.

44. We might have found the ſame reſults without any geometrical conſtruction by means of the formulæ already given, as follows. Let S be put for the ſecant of A, and *s* for that of B: then by the known properties of *ſines*, *coſines*, *tangents* and *ſecants*, we ſhall have; *ſin.* A $= \frac{Tr}{S}$, and, *coſ.* A $= \frac{rr}{S}$; *ſin.* B $= \frac{tr}{s}$, and, *coſ.* B $= \frac{rr}{s}$; which values being ſubſtituted in thoſe of, *ſin.* $\overline{A \pm B}$ and *coſ.* $\overline{A \pm B}$, we ſhall get, *ſin.* $\overline{A+B} = \frac{r^2 \times \overline{T+t}}{Ss}$, and, *ſin.* $\overline{A-B} = \frac{r^2 \times \overline{T-t}}{Ss}$; *coſ.* $\overline{A+B} = \frac{r \times \overline{r^2 - Tt}}{Ss}$, and, *coſ.* $\overline{A-B} = \frac{r \times \overline{rr+Tt}}{Ss}$; and therefore, ſince *tang.* $= \frac{r \times \mathit{ſin.}}{\mathit{coſ.}}$, we ſhall find as before, *tang.* $\overline{A+B} = \frac{r^2 \times \overline{T+t}}{rr - Tt}$, and, *tang.* $\overline{A-B} = \frac{r^2 \times \overline{T-t}}{rr+Tt}$.

COROLLARY I.

45. $\mathit{Tang.}\ \overline{A+B} \times \mathit{tang.}\ \overline{A-B} =$

$$\frac{\overline{\mathit{tang.}\,A + \mathit{tang.}\,B} \times \overline{\mathit{tang.}\,A - \mathit{tang.}\,B} \times r^4}{\overline{rr - \mathit{tang.}\,A \times \mathit{tang.}\,B} \times \overline{rr + \mathit{tang.}\,A \times \mathit{tang.}\,B}} = \frac{\overline{\mathit{tang.}^2 A - \mathit{tang.}^2 B} \times r^4}{r^4 - \mathit{tang.}^2 A \times \mathit{tang.}^2 B}.$$

COROLLARY II.

46. If one of the angles be 45°, we ſhall have, $\mathit{tang.}\ \overline{A+45^\circ} = \frac{\overline{1+\mathit{tang.}\,A} \times r}{1-\mathit{tang.}\,A} = \frac{rr}{\mathit{tang.}\ \overline{45^\circ - A}}$;

becauſe the radius is (by *art.* 12) a mean proportional between *tang.* $\overline{45^\circ + A}$ and *tang.* $\overline{45^\circ - A}$: from whence it follows, that if we calculate the *tangents* of all the arcs below 45°, we ſhall get the *tangents* of all the arcs above 45°, by ſimple diviſions; and, if we compute the logarithms of the ſame *tangents*, we ſhall likewiſe get the logarithms of the *tangents* of all the arcs above 45°, by ſubtracting the logarithms of the *tangents* below 45° from double the logarithm of the ſine total.——This Corollary therefore ſhews an eaſy method of conſtructing the Tables of *natural* and *artificial* tangents.

COROLLARY III.

47. Whilſt the arcs A and B continue ſuch, that their ſum is leſs than 90°, the expreſſion of *tang.* $\overline{A+B}$ will be poſitive: if we ſuppoſe *tang.* A × *tang.* B to be $= rr$, (which happens when theſe arcs are the complements to each other), the denominator then becomes nothing, and conſequently the *tangent* infinite; but when *tang.* A × *tang.* B is greater than rr, (which is the caſe when the ſaid arcs taken together conſtitute an arc greater than 90°), the expreſſion of the *tangent* then becomes negative, and of conſequence muſt be taken in an oppoſite ſenſe†.

COROLLARY IV.

48. If we ſuppoſe the arcs A and B to be equal, and imagine a ſeries of arcs, A, 2A, 3A, 4A, &c. multiples of the firſt; it will be eaſy to form the following Table of their *tangents* from the formu-

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† See *Maſeres*'s Elements of Trigonometry, p. 78, &c. or, *Simpſon*'s Trigonometry, p. 56.

la, *Tang.* $\overline{A+B} = \frac{r^2 \times \overline{tang.\ A + tang.\ B}}{r^2 - tang.\ A \times tang.\ B}$; by always regarding the *tangent* laſt found as *tang.* A, and making B equal to the ſimple arc A, whereof we ſeek the multiples,

$$Tang.\ A = T.$$

$$Tang.\ 2A = \frac{2r^2T}{rr - TT}.$$

$$Tang.\ 3A = \frac{8rrT - T^3}{r^2 - 3TT}.$$

$$Tang.\ 4A = \frac{4r^4T - 4r^2T^3}{r^4 - 6r^2T^2 + T^4}.$$

$$Tang.\ 5A = \frac{5r^4T - 10r^2T^3 + T^5}{r^4 - 10r^2T^2 + 5T^4}.$$

And from hence it will likewiſe be eaſy to deduce the following general formula, by obſerving the law of the coefficients for any multiple arc, and making the radius equal to unity.

$$Tang.\ nA =$$

$$\frac{nT - \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\ 2.\ 3}T^3 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5}T^5 - \&c.}{1 - \frac{n \times \overline{n-1}}{1.\ 2}T^2 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.\ 2.\ 3.\ 4}T^4 - \&c.}.$$

But in order to abridge this formula, (as well as all thoſe which have been hitherto given), we may obſerve, that every ſucceeding coefficient contains that of the term which precedes it; and therefore, if the coefficients of the numerator be repreſented by the indeterminate letters, A, B, C, D, &c. and thoſe of the denominator by, α, β, γ, δ, &c. the foregoing formula may be expreſſed thus;

$$Tang.\ nA =$$

$$\frac{nT - A.\frac{\overline{n-1}.\overline{n-2}}{2.\ 3}T^3 + B.\frac{\overline{n-3}.\overline{n-4}}{4.\ 5}T^5 - C.\frac{\overline{n-5}.\overline{n-6}}{6.\ 7}T^7 + \&c.}{1 - \frac{n \times \overline{n-1}}{1.\ 2}T^2 + \alpha.\frac{\overline{n-2}.\overline{n-3}}{3.\ 4}T^4 - \beta.\frac{\overline{n-4}.\overline{n-5}}{5.\ 6}T^6 + \&c.}.$$

49. If we would expreſs the *tangents* of multiple arcs by beginning at the higheſt terms or higheſt powers of T, we ſhall readily perceive, that two general ſeries or formulæ will be requiſite; the one for odd numbers, and the other for even ones; and that, ſince the higheſt odd powers are alternately poſitive and negative, the multiple *tangent* muſt have the two ſigns ± prefixed to it.

Firſt formula for n *when odd.*

$$\pm\,\mathit{Tang.}\,nA = \frac{T^{n} - \frac{n.\overline{n-1}}{1.2}T^{n-2} + A.\frac{\overline{n-2}.\overline{n-3}}{3.4}T^{n-4} - B.\frac{\overline{n-4}.\overline{n-5}}{5.6}T^{n-6} + \&c.}{nT^{n-1} - \alpha.\frac{\overline{n-1}.\overline{n-2}}{2.3}T^{n-3} + \beta.\frac{\overline{n-3}.\overline{n-4}}{4.5}T^{n-5} - \gamma.\frac{\overline{n-5}.\overline{n-6}}{6.7}T^{n-7} + \&c.}$$

Second formula for n *when any even number:*

$$\pm\,\mathit{Tang.}\,nA = \frac{nT^{n-1} - A\frac{\overline{n-1}.\overline{n-2}}{2.3}T^{n-3} + B.\frac{\overline{n-3}.\overline{n-4}}{4.5}T^{n-5} - \&c.}{T^{n} - \frac{n.\overline{n-1}}{1.2}T^{n-2} + \alpha.\frac{\overline{n-2}.\overline{n-3}}{3.4}T^{n-4} - \&c.}$$

50. If we would have the *cotangents* concerned in the expreſſions of the *tangents* of multiple arcs, we have nothing more to do, than to ſubſtitute *cot.* in the place of $\frac{RR}{T}$; after which, it will be eaſy to change the preceding Table into the following one for the *cotangents* of multiple arcs.

$$Cot.\ A = cot.\ A.$$

$$Cot.\ 2\,A = \frac{cot.\ A - tang.\ A}{2}$$

$$Cot.\ 3\,A = \frac{r^2\ cot.\ A - 3r^2\ tang.\ A}{3\,r^2 - tang^2.\ A}$$

$$Cot.\ 4\,A = \frac{r^2\ cot.\ A - 6\,r^2\ tang.\,A + tang.^3\,A}{4\,r^2 - 4\ tang^2.\ A}$$

$$Cot.\ 5\,A = \frac{r^2\ cot.\ A - 10\,r^2\ tang.\ A + 5\ tang.^3\,A}{5r^2 - 10\ tang^2.\ A + tang^4.\ A}$$

$$Cot.\ 6\,A = \frac{r^4\ cot.\,A - 15\ r^4 tang.\ A + 15 r^2 tang^3.\,A - tang.^5\,A}{6r^4 - 20r^2\ tang^2.\ A + 6\,tang^4.\,A}$$

And in general we ſhall have;

$$Cot.\ n\,A =$$

$$\frac{cot.\ A - \frac{n.\,\overline{n-1}}{1.\ 2} tang.\ A + A.\frac{\overline{n-2}.\,\overline{n-3}}{3.\ 4} tang.^3\,A - \&c.}{n - \alpha.\frac{\overline{n-1}.\overline{n-2}}{2.\ 3} tang.^2 A + \beta.\frac{\overline{n-3}.\overline{n-4}}{4.\ 5} tang.^4 A - \&c.}$$

by making the radius equal to unity; becauſe by ſpecifying the radius, two ſeries would be neceſſary for the *cotangents* of multiple arcs, according as *n* ſhould be an even or odd number.

51. The two laſt general formulæ might likewiſe, it is manifeſt, be applied to the *cotangents* of multiple arcs, by beginning at the terms of the higheſt powers, ſince the coefficients are always the ſame. We may alſo make uſe of them, to transform the powers of *tangents* into *tangents* of the ſimple arc and its multiples: thus, if we ſuppoſe, T′, T″, T‴, &c. to denote the *tangents* of multiple arcs, we ſhall eaſily obtain the following Table by ſimple ſubſtitutions.

$$T=T.$$

$$T^2=\frac{T''-2T}{T''}.$$

$$T^3=\frac{3T''T-6T'''T+2T'''T''}{T''}.$$

$$T^4=\frac{-8TT''+24TT'''-8T''T'''-12TT^{IV}+5T'T^{IV}}{T''T^{IV}}.$$

&c.

COROLLARY.

52. It follows from hence, that in order to obtain all the *tangents* from 0° to 90°, it will be ſufficient to find them to 30°: for if any arc leſs than 30° be called B; and we have, $A=30^\circ-B$; we ſhall alſo have, $2A=60^\circ-2B$, and, *cot.* $2A=$ *tang.* $\overline{30^\circ+2B}$; and therefore, on account of the formula (*art.* 20), *cot.* $2A=\frac{1}{2}$*cot.*$A-\frac{1}{2}$ *tang.* A, we ſhall get, *tang.*$\overline{30^\circ+2B}=\frac{1}{2}$*cot.*$\overline{30^\circ-B}-\frac{1}{2}$*tang.*$\overline{30^\circ-B}$: from whence it eaſily appears, how the calculation of the Tables of *tangents* may be as much ſhortened, as we have already ſhewn that the calculation of thoſe of the *ſines* and *coſines* might be.

FIRST SCHOLIUM.

53. By the formulæ, which we have given in the preceding general ſeries, we may not only find the *ſines*, *coſines*, *tangents* and *cotangents* of multiple arcs, but alſo the *ſine*, *coſine*, *&c.* of any ſubmultiple arc or angle in any relation whatever; but then, it is manifeſt, we ſhall have equations of different dimenſions concerned; which, when n is a whole or commenſurable number, will be always finite, and admit of eaſy ſolutions by means of the Tables of *ſines*, &c. For example, if we would divide an arc into three or five equal parts, by means of its *ſine* or *coſine*, we need only make, *ſin.* $nA=a$, and, *coſ.* $nA=b$; $s=x$, and, $c=y$; and the formulæ

B and D will give the following equations, $4x^3 - 3r^2x + ar^2 = 0$, and, $4y^3 - 3r^2y - br^2 = 0$. In like manner, ſuppoſing n ſtill $= 3$; *tang.* n A $= c$, and, *cot.* n A $= d$; *tang.*A $= z$, and, *cot.* A $= u$; the formulæ in *art.* 48 and 50 give, $z^3 - 3cz^2 - 3r^2z + r^2c = 0$, and, $u^3 - 3du^2 - 3r^2u + r^2d = 0$: from whence it follows, that the triſection of an angle always produces an equation of three dimenſions, in whatever manner its ſolution be attempted.

SECOND SCHOLIUM.

Fig. 7. 54. But, that we may be able to form clear ideas of the changes whereof the preceding formulæ are ſuſceptible with regard to the ſigns; let us conceive a radius CA to remain fixed, whilſt a moveable radius C M, by deſcribing the circumference AB*ab*A, gives the ſeveral arcs AM, A*m*, Aμ, &c. and it will appear, by taking the point A for the common origin of all theſe arcs, as well as of their *tangents*, that all the poſſible changes relating to the *ſines*, *coſines*, *tangents* and *cotangents*, are reduced to the following:

1°. The *ſine* of the naſcent arc hath 0 for its limit, and its *coſine* equal to radius; therefore, ſince *tang.* $= \frac{\mathrm{R} \times \textit{ſin.}}{\textit{coſ.}}$, we ſhall have *tang.* $= 0$, and *cot.* $= \infty$. 2°. The *ſine* increaſes, and *coſine* decreaſes, from 0 to 90°; therefore the *tang.* increaſes, and *cot.* decreaſes, poſitively. 3°. At 90°, *ſin.* $=$ R, and *coſ.* $= 0$; therefore *tang.* $= \infty$, and *cot.* $= 0$. 4°. From this point to 180°, the *ſine* always decreaſes poſitively, and the *coſine* increaſes negatively; therefore the *tang.* decreaſes, and *cot.* increaſes, negatively. 5°. At 180°, *ſin.* $= 0$, and *coſ.* $= -$R; therefore *tang.* $= 0$, and *cot.* $= -\infty$. 6°. After 180°, the *ſine* increaſes, and *coſine* decreaſes, negatively; therefore the *tang.* increaſes, and *cot.* decreaſes, poſitively.

7°.

7°. At 270°. *ſin.* $=-$R, and *coſ.* $=0$; therefore *tang.* $=-\infty$, and *cot.* $=0$. 8°. Laſtly, after 270°, the *ſine* decreaſes negatively, and the *coſine* increaſes poſitively; therefore, the *tang.* decreaſes and *cot.* increaſes, negatively to 360°, where the whole becomes as in *art.* 1.——— As in trigonometrical calculations only right, obtuſe, or acute angles are made uſe of, all the conſiderations of the ſigns are reduc'd to inquire in what caſes the formulæ, which we have given, indicate an obtuſe or acute angle: and this is very ſimple; for according to our ſuppoſitions, whenever the expreſſion of a *coſine* or *cotangent* is found to be negative, we are then to regard the angle, to which theſe ſigns belong, as obtuſe.

Preparation to the following Theorems.

55. Let PE be the arc which we have denoted by A, and PB that which we have called B. From the point B let there be drawn the lines BK and BH reſpectively perpendicular to the lines DE and CP. Moreover, having drawn the lines BE and BO to the extremities of the chord EO, let fall upon theſe lines the perpendiculars CT and CR, terminated by the tangent RBT. This conſtruction being perfectly underſtood, it will appear; 1°. that, the arc BO $=$ A $+$ B; 2°. that, BE $=$ A $-$ B; 3°. that, KO $=$ *ſin.* A $+$ *ſin.* B; 4°. that, KE $=$ *ſin.* A $-$ *ſin.* B; 5°. that, KM $=$ *coſ.* B $+$ *coſ.* A; 6°. that, KB $=$ *coſ.* B $-$ *coſ.* A; 7°. that, BT $=$ *tang.* $\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$; 8°. that, BR $=$ *tang.* $\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$; 9°. that, AL $=$ *ſin.* $\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$; 10°. that, CL $=$ *coſ.* $\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$; 11°. that, BF $=$ *ſin.* $\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$; 12°. that, CF $=$ *coſ.* $\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$; 13°. that, OM $=2$ CF $=2$ *coſ.* $\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$; (for the arc MO $=$ MN $+$ NO; but *Fig.* 8.

but, NO = 180°—A, and, MN = B; therefore $\frac{MO}{2}$ = ſin. $\overline{90^\circ - \frac{1}{2}A + \frac{1}{2}B}$; and conſequently, MO = 2 coſ. $\overline{\frac{1}{2}A - \frac{1}{2}B}$; and that in like manner, ME = 2 CL = 2 coſ. $\overline{\frac{1}{2}A + \frac{1}{2}B}$. So much being premiſed, the following Theorems will be obtained and underſtood without any difficulty.

THEOREM VII.

56. *Suppoſing all things as in the preceding conſtruction, I ſay, that we ſhall have*, 1°, *Sin.* A + *ſin.* B = 2 *ſin.* $\overline{\frac{1}{2}A + \frac{1}{2}B} \times$ *coſ.* $\overline{\frac{1}{2}A - \frac{1}{2}B}$, and 2°, *Sin.* A — *ſin.* B = 2 *ſin.* $\overline{\frac{1}{2}A - \frac{1}{2}B} \times$ *coſ.* $\overline{\frac{1}{2}A + \frac{1}{2}B}$.

DEMONSTRATION.

The right-angled triangles, MKO, CLA, are manifeſtly ſimilar, ſince the angle at M in the former is equal to that at C in the latter; and therefore we ſhall have, OK : OM :: AL : AC; or, by ſubſtituting for theſe lines their expreſſions in *ſines* and *coſines*, *Sin.* A + *ſin.* B : 2 *coſ.* $\overline{\frac{1}{2}A - \frac{1}{2}B}$ :: *ſin.* $\overline{\frac{1}{2}A + \frac{1}{2}B}$: R; from whence, by making the radius equal to unity, the firſt part of the Theorem is deduced. Q. E. 1°. D.

2°. On account of the ſimilar triangles, EKB, CLA, we ſhall alſo have, EK : EB :: CL : CA; or, *Sin.* A — *ſin.* B : 2 *ſin.* $\overline{\frac{1}{2}A - \frac{1}{2}B}$ :: *coſ.* $\overline{\frac{1}{2}A + \frac{1}{2}B}$: R; from whence we deduce the ſecond part of the Theorem. Q. E. 2°. D.

THEOREM VIII.

57. *I ſay moreover, that we ſhall have the two following equations*; 1°, *Coſ.* A + *coſ.* B = 2 *coſ.* $\overline{\frac{1}{2}A + \frac{1}{2}B} \times$ *coſ.* $\overline{\frac{1}{2}A - \frac{1}{2}B}$; 2°, *Coſ.* B — *coſ.* A = 2 *ſin.* $\overline{\frac{1}{2}A + \frac{1}{2}B} \times$ *ſin.* $\overline{\frac{1}{2}A - \frac{1}{2}B}$.

DEMONSTRATION.

The ſame ſimilar triangles, MKO, CLA and EKB, give yet the two following analogies; MK : MO : : CL : CA, and, BK : BE : : AL : CA; which, by ſubſtituting the particular value of each term, will be thus expreſſed: $\textit{Coſ.}\,A + \textit{coſ.}\,B : 2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B} :: \textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} : R$, and, $\textit{Coſ.}\,B - \textit{coſ.}\,A : 2\,\textit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B} :: \textit{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} : R$: from whence are immediately deduced the two equations which were required to be demonſtrated.

COROLLARY I.

58. Hence, it is manifeſt that, we ſhall get,
$$\frac{\textit{ſin.}\,A + \textit{ſin.}\,B}{\textit{ſin.}\,A - \textit{ſin.}\,B} = \frac{2\,\textit{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}{2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}} = \frac{\textit{tang.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}}{\textit{tang.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}$$
$= \textit{tang.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{cot.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$, by ſubſtituting *tang.* for $\frac{\textit{ſin.}}{\textit{coſ.}}$ &c.

COROLLARY II.

59. Alſo, $\frac{\textit{ſin.}A + \textit{ſin.}B}{\textit{coſ.}A + \textit{coſ.}B} = \frac{2\,\textit{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}{2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}$ $= \textit{tang.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$, by expunging the quantities that deſtroy, and putting *tang.* for $\frac{\textit{ſin.}}{\textit{coſ.}}$

COROLLARY III.

60. We ſhall moreover find that, $\frac{\textit{ſin.}A - \textit{ſin.}B}{\textit{coſ.}A + \textit{coſ.}B} =$ $\frac{2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}{2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}} = \textit{tang.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$; that
$$\frac{\textit{ſin.}\,A - \textit{ſin.}\,B}{\textit{coſ.}\,B - \textit{coſ.}\,A} = \frac{2\,\textit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}{2\,\textit{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \textit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}} = \textit{cot.}$$

$\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$, and that, $\dfrac{\text{coſ.}\,B + \text{coſ.}\,A}{\text{coſ.}\,B - \text{coſ.}\,A} = \dfrac{2\,\text{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}{2\,\text{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}} = \dfrac{\text{cot.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}}{\text{tang.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}} = \text{cot.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{cot.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.

COROLLARY IV.

61. We might likewiſe find by ſimple ſubſtitutions (though more immediately from what hath been demonſtrated in *art.* 13. $R : \text{coſ.}\,\tfrac{1}{2}A :: 2\,\text{ſin.}\,\tfrac{1}{2}A : \text{ſin.}\,A$, by ſuppoſing $A = A \pm B$), that, $\dfrac{\text{ſin.}\,\overline{A+B}}{\text{ſin.}\,\overline{A-B}} = \dfrac{\text{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}}{\text{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}$, and alſo that, $\dfrac{\text{coſ.}\,\overline{A+B}}{\text{coſ.}\,\overline{A-B}} = \dfrac{R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}}{R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}$. Moreover, we might yet deduce from the different ſimilar triangles, which occur in *fig.* 8, a great number of other properties beſides thoſe we have here ſpecified; as the formulæ which are given in *art.* 26, for inſtance; but what we have already ſhewn will, we preſume, be quite ſufficient for pointing out the method of obtaining ſuch properties as we may at any time chance to have occaſion for; and therefore it only now remains, that we make ſome applications of the preceding formulæ to the logarithms.

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* Since $R : \text{coſ.}\,\tfrac{1}{2}A :: 2\,\text{ſin.}\,\tfrac{1}{2}A : \text{ſin.}\,A$, we ſhall have, $R^2 : \text{coſ.}^2\,\tfrac{1}{2}A$ or $R^2 - \text{ſin.}^2\,\tfrac{1}{2}A :: 4\,\text{ſin.}^2\,\tfrac{1}{2}A : \text{ſin.}^2\,A$ or $R^2 - \text{coſ.}^2\,A$; and therefore, $R^4 - R^2 \times \text{coſ.}^2\,A = 4R^2 \times \text{ſin.}^2\,\tfrac{1}{2}A - 4\,\text{ſin.}^4\,\tfrac{1}{2}A$, or, $R^2 \times \text{coſ.}^2\,A = R^4 - 4R^2 \times \text{ſin.}^2\,\tfrac{1}{2}A + 4\,\text{ſin.}^4\,\tfrac{1}{2}A$; whence, $R \times \text{coſ.}\,A = R^2 - 2\,\text{ſin.}^2\,\tfrac{1}{2}A$, or, $R \times \text{coſ.}\,\overline{A \pm B} = R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A \pm \tfrac{1}{2}B}$, and conſequently, $\dfrac{\text{coſ.}\,\overline{A+B}}{\text{coſ.}\,\overline{A-B}} = \dfrac{R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}}{R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}}$.

Uſe of ſome of the preceding Formulæ in the Logarithms.

62. The formulæ, $\text{ſin. } A + \text{ſin. } B = 2\,\text{ſin. } \overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{coſ. } \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$ and $\text{ſin. } A - \text{ſin. } B = 2\,\text{coſ. } \overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{ſin. } \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$ furniſh us with the means of obtaining the logarithms of the ſum and difference of two given quantities.——An example or two will convey a ſufficiently clear idea of the method of operation.

EXAMPLE I.

63. Let the numbers 8467 and 8635 be propoſed, and let it be required to find, from the formula, $\text{ſin. } A + \text{ſin. } B = 2\,\text{ſin. } \overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \text{coſ. } \overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$, the logarithm of their ſum; which it is imagined cannot be found amongſt thoſe of the Tables. (The proceſs would be exactly ſimilar with reſpect to any two other numbers, even though they ſhould contain fractions, provided their particular logarithms were given.) Now the firſt thing to be performed, in this example, is to obtain the values of the angles A and B; and to do this, I add 6 to each of the characteriſtics of the propoſed numbers (that I may be able to find the ſaid angles in the Tables of the logarithms of the *ſines*, becauſe theſe logarithms have no characteriſtic below 7 or above 9), and I thereby get

$9.936262 = \text{log. } 8635 = \text{log. ſin. } 59^\circ\ 42'\ 42'' = A.$

$9.927730 = \text{log. } 8467 = \text{log. ſin. } 57^\circ\ 51'\ 16'' = B;$

and of conſequence deduce, $\tfrac{1}{2}A + \tfrac{1}{2}B = 58^\circ\ 46'\ 59''$; and $\tfrac{1}{2}A - \tfrac{1}{2}B = 0^\circ\ 55'\ 43''$.

This done, I, in the next place, ſeek the logarithms of the ſine of the firſt of the laſt angles, and that of the coſine of the latter of them, and to the ſum of theſe logarithms add the logarithm of the number 2; then from this ſum I ſubtract 10 + 6, or 16 (on account of the radius which divides, and the ſix units that were before added to the characteriſtics), and find that the logarithm of 17102 is equal to the remainder 4.233046.

9.932078 = log. ſin. $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$.
9.999943 = log. coſ. $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.
0.301030 = log. 2.
20.233046
4.233046 = log. 17102.

EXAMPLE II.

64. The ſame formulæ might likewiſe be made uſe of in order to conſtruct by the logarithms an expreſſion more complicated than the preceding; ſuch as, for inſtance, this, $r \times \frac{AA+aa-bb}{2Aa}$; wherein I ſhall ſuppoſe r to repreſent the ſine total; that A is a number whoſe logarithm = 4.054723; that the logarithm of a = 3.108354, and that of b = 3.876870; and conſequently that the logarithms of the quantities, A^2, a^2 and b^2 = the doubles of theſe. Thus much being ſuppoſed, I firſt regard the logarithms of A^2 and a^2 as thoſe of the ſines of two angles, whoſe ſum is to be found: then I add unity to each of their characteriſtics (that I may be enabled to find their logarithms among thoſe of the ſines), and, having found the angles correſponding to the reſulting ſums, perform the opera-

tion in the manner annexed; and ſubtracting 10 + 1 from the characteriſtic of the number laſt found, I get the logarithm of A + a = 8.114974; then conſidering this logarithm as that of *ſin.* A, and adding unity thereto as well as to the logarithm of *b*, I complete the operation as follows, by means of the formula, *ſin.* A — *ſin.* B = 2 *ſin.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$ × *coſ.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$;

log. A = 9.109446 = *log. ſin.* 7° 23′ 32″ = A.
log. a = 7.216708 = *log. ſin.* 0° 5′ 40″ = B.
therefore $\tfrac{1}{2}$A + $\tfrac{1}{2}$B = 3° 44′ 36″.
and $\tfrac{1}{2}$A — $\tfrac{1}{2}$B = 3° 38′ 56″.

0.301030 = *log.* 2.
8.814825 = *log. ſin.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$
9.999119 = *log. coſ.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$
19.114974

9.114974 = *log. ſin.* 7° 29′ 14″.
8.753740 = *log. ſin.* 3° 15′ 6″.
therefore $\tfrac{1}{2}$A + $\tfrac{1}{2}$B = 5° 22′ 10″.
$\tfrac{1}{2}$A — $\tfrac{1}{2}$B = 2° 7′ 4″.

0.301030 = *log.* 2.
8.567658 = *log. ſin.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$
9.998090 = *log. coſ.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$
18.866778
7.866778 = *log.* $\overline{A^2 + a^2 - b^2}$.

0.301030 = *log.* 2.
4.054723 = *log.* A.
3.108354 = *log.* a.
7.464107 = *log.* 2 A a.

17.866778 = *log.* $\overline{A^2 + a^2 - b^2}$
7.464107 = *log.* 2 A a.
10.402671 = *log.* $\dfrac{A^2 + a^2 - b^2}{2Aa} \times r$.

Of the Uſe of the arithmetical Complement.

65. Before I put an end to theſe obſervations, it will not be improper to aſcertain the meaning and give a preciſe notion of what is called the *arithmetical Complement*, which Geometers very frequently uſe to

convert ſubtraction into addition.—And that this may be done, let it be required to ſubtract 754 from 896. Now I eaſily perceive, that if I take the ſubtrahend, 754, from 1000, add the difference to 896, and from the ſum thence ariſing take the figure which is found in the place of thouſands, the ſame remainder will be left as if I ſubtract it from 896, by the common method: but, in order to ſubtract this number, 754, from 1000, it will be ſufficient to put down ſuch figures as make 9 with each of thoſe of the ſubtrahend, the laſt figure excepted, which muſt always make 10, as is proved and demonſtrated by ſubtraction; and the number reſulting from this operation, that is to ſay, (in the preſent caſe) 246 is the *arithmetical Complement*; conſequently, if I add this number to 896, and afterwards from their ſum, 1142, take the unit which poſſeſſes the place of thouſands, I ſhall get 142, for the true remainder ſought.

COROLLARY.

66. From the preceding reaſoning it follows, that the logarithms of the formulæ, $cot.\ A = \frac{RR}{tang.\ A}$; $tang.\ A = \frac{RR}{cot.\ A}$; $coſec.\ A = \frac{RR}{ſin.\ A}$ &c. will be found by adding the arithmetical Complements of the denominators to the logarithm of the radius; or, which is the ſame, by ſubtracting the logarithms of the denominators from the logarithm of the ſquare of the radius, which hath 20 for its index.

SCHOLIUM.

67. As it may be ſometimes neceſſary to make different combinations or ſubſtitutions of the formulæ which we have demonſtrated in this Chapter,

and as it would frequently prove inconvenient to be obliged to ſeek for them in the different articles wherein they are ſpecified, we hope that our readers will not be diſpleaſed to find them all collected together and united under one view; particularly, as ſuch an arrangement will ſerve to refer to, whenever, in the following parts of this work, we ſhall have any ſubſtitution to make. We have taken the greateſt care poſſible to preſerve throughout the homogeneity of the terms, that we may be enabled to apply theſe formulæ, as occaſion ſhall require, with the utmoſt preciſion, and the leaſt danger of miſtake.

A general Table of the Formulæ demonſtrated in this Chapter.

68. $\text{Sin. } \frac{1}{2} \text{A} = \text{coſ. } \frac{1}{2} \text{ſup. A}; \text{ tang. } \frac{1}{2} \text{A} = \text{cot. } \frac{1}{2} \text{ſup. A}$ (*art.* 7).

69. $\frac{\text{Sin. A} \times \text{R}}{\text{coſ. A}} = \text{tang. A}$ (*art.* 9) $= \frac{\text{ſin. A} \times \text{ſec. A}}{\text{R}}$ (*art.* 15).

70. $\frac{\text{Coſ. A} \times \text{R}}{\text{ſin. A}} = \text{cot. A}$ (*art.* 10) $= \frac{\text{RR}}{\text{tang. A}}$ (*art.* 12).

71. $\text{Sin. } 2\text{A} = \frac{\text{coſ. A} \times 2 \text{ ſin. A}}{\text{R}}$ (*art.* 13); and, $\text{coſ. } 2\text{A} = \frac{2 \text{ coſ.}^2 \text{A} - \text{RR}}{\text{R}}$.

72. $\frac{\text{Sin. } 2\text{A}}{2 \text{ ſin. A}} = \frac{\text{coſ. A}}{\text{R}} = \frac{\text{RR} - 2 \text{ ſin.}^2 \frac{1}{2}\text{A}}{\text{RR}}$ (*art.* 21).

73. $\frac{1}{2} \text{Sin. } 2\text{A} = \frac{\text{coſ. A} \times \text{ſin. A}}{\text{R}} = \frac{\text{ſin.}^2 \text{A} \times \text{cot. A}}{\text{RR}} = \frac{\text{ſin.}^2 \text{A}}{\text{tang. A}}$ (*art.* 14).

74. $\text{R} + \text{coſ. A} = \frac{2 \text{ coſ.}^2 \frac{1}{2}\text{A}}{\text{R}} = \frac{\text{ſin. A} \times \text{R}}{\text{tang. } \frac{1}{2}\text{A}}$ (*art.* 21) $= \text{ſin. v. ſup. A}$ (*art.* 22); $\text{R} + \text{ſin. A} = 2 \text{ ſin.}^2 \, 45^\circ + \frac{1}{2}\text{A}$.

75. $R - \text{coſ.}\,A = \frac{2\,\text{ſin.}^2\,\frac{1}{2}A}{R} = \frac{\text{ſin.}\,A \times \text{tang.}\,\frac{1}{2}A}{R}$ (*art.* 21.) $= \text{ſin. V.}\,A$ (*art.* 22): $R - \text{ſin.}\,A = 2\,\text{ſin.}^2\,\overline{45^\circ - \frac{1}{2}A}$.

76. $\frac{R + \text{coſ.}\,A}{R - \text{coſ.}\,A} = \frac{\text{cot.}^2\,\frac{1}{2}A}{RR}$: $\frac{R - \text{coſ.}\,A}{R + \text{coſ.}\,A} = \frac{\text{tang}^2\,\frac{1}{2}A}{RR}$ (*art.* 22).

77. $\frac{R + \text{ſin.}\,A}{R - \text{ſin.}\,A} = \frac{\text{ſin.}^2\,\overline{45^\circ + \frac{1}{2}A}}{\text{ſin.}^2\,\overline{45^\circ - \frac{1}{2}A}}$ (d°).

78. $\text{Tang.}\,A = \frac{RR - \text{tang.}^2\,\frac{1}{2}\,\text{comp.}\,A}{2\,\text{tang.}\,\frac{1}{2}\,\text{comp.}\,A}$ (*art.* 20).

79. $\text{Cot.}\,A = \frac{RR - \text{tang.}^2\,\frac{1}{2}A}{2\,\text{tang.}\,\frac{1}{2}A}$ (d°).

80. $\text{Sec.}\,A = \frac{RR}{\text{coſ.}\,A} = \frac{\text{tang.}\,A \times R}{\text{ſin.}\,A.}$ (*art.*15 and 69).

81. $\text{Coſec.}\,A = \frac{RR}{\text{ſin.}\,A}$ (*art.* 15).

82. $\frac{\text{Sec.}\,A \times R}{\text{coſec.}\,A} = \frac{\text{ſin.}\,A \times R}{\text{coſ.}\,A} = \text{tang.}\,A$ (*art.* 16).

83. $\text{Sec.}\,A = \text{cot.}\,\frac{1}{2}\,\text{comp.}\,A - \text{tang.}\,A$ (*art.* 18) $= \text{tang.}\,A + \text{tang.}\,\frac{1}{2}\,\text{comp.}\,A$ (*art* 19) $= \frac{\text{cot.}\,\overline{45^\circ - \frac{1}{2}A} + \text{tang.}\,\overline{45^\circ - \frac{1}{2}A}}{2} = \frac{\text{tang.}\,\overline{45^\circ + \frac{1}{2}A} + \text{tang.}\,\overline{45^\circ - \frac{1}{2}A}}{2}$

84. $\text{Coſec.}\,A = \text{cot.}\,\frac{1}{2}A - \text{cot.}\,A$ (*art.* 18) $= \text{cot.}\,A + \text{tang.}\,\frac{1}{2}A$ (*art.* 19) $= \frac{\text{cot.}\,\frac{1}{2}A + \text{tang.}\,\frac{1}{2}A}{2}$.

85. $\text{Sin.}\,\overline{A \pm B} = \frac{\text{ſin.}\,A \times \text{coſ.}\,B \pm \text{coſ.}\,A \times \text{ſin.}\,B}{R}$ (*art.* 23).

86. $\text{Coſ.}\,\overline{A \pm B} = \frac{\text{coſ.}\,A \times \text{coſ.}\,B \mp \text{ſin.}\,A \times \text{ſin.}\,B}{R}$ (*art.* 24).

87. $\frac{\text{Sin.}\,\overline{A + B}}{\text{Sin.}\,\overline{A - B}} = \frac{\text{tang.}\,A + \text{tang.}\,B}{\text{tang.}\,A - \text{tang.}\,B}$ (*art.* 25) $= \frac{2\,\text{ſin.}\,\overline{\frac{1}{2}A + \frac{1}{2}B} \times \text{coſ.}\,\overline{\frac{1}{2}A + \frac{1}{2}B}}{2\,\text{ſin.}\,\overline{\frac{1}{2}A - \frac{1}{2}B} \times \text{coſ.}\,\overline{\frac{1}{2}A - \frac{1}{2}B}}$ (*art.* 61).

88. $\frac{\text{Coſ.}\,\overline{A+B}}{\text{Coſ.}\,\overline{A-B}} = \frac{\text{cot.}\,B - \text{tang.}\,A}{\text{cot.}\,B + \text{tang.}\,A}$ (art. 25) $=$

$\frac{\overline{R+\sqrt{2}\,\text{ſin.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}} \times \overline{R-\sqrt{2}\,\text{ſin.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}}}{\overline{R+\sqrt{2}\,\text{ſin.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}} \times \overline{R-\sqrt{2}\,\text{ſin.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}}} = \frac{R^2 - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}}{R - 2\,\text{ſin.}^2\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}}$ (art. 61).

89. $\text{Sin.}\,\overline{30^\circ+A} = \text{coſ.}\,A - \text{ſin.}\,\overline{30^\circ-A}$ (art. 27).

90. $\text{Coſ.}\,\overline{30^\circ+A} = \text{coſ.}\,\overline{30^\circ-A} - \text{ſin.}\,A$ (d°).

91. $\text{Sin.}\,A + \text{ſin.}\,B = 2\,\text{ſin.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ (art. 56).

92. $\text{Sin.}\,A - \text{ſin.}\,B = 2\,\text{ſin.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$ (d°).

93. $\text{Coſ.}\,A + \text{coſ.}\,B = 2\,\text{coſ.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B} \times \text{coſ.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ (art. 57).

94. $\text{Coſ.}\,B - \text{coſ.}\,A = 2\,\text{ſin.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B} \times \text{ſin.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ (d°).

95. $\frac{\text{Sin.}\,A+\text{ſin.}\,B}{\text{Sin.}\,A-\text{ſin.}\,B} = \frac{\text{tang.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}}{\text{tang.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}}$ (art. 58).

96. $\frac{\text{Sin.}\,A+\text{ſin.}\,B}{\text{Coſ.}\,A+\text{coſ.}\,B} = \text{tang.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$ (art. 59).

97. $\frac{\text{Sin.}\,A+\text{ſin.}\,B}{\text{Coſ.}\,B-\text{coſ.}\,A} = \text{cot.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ (art. 56 and 57).

98. $\frac{\text{Sin.}\,A-\text{ſin.}\,B}{\text{Coſ.}\,B+\text{coſ.}\,A} = \text{tang.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}$ (art. 60).

99. $\frac{\text{Sin.}\,A-\text{ſin.}\,B}{\text{Coſ.}\,B-\text{coſ.}\,A} = \text{cot.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}$ (d°).

100. $\frac{\text{Coſ.}\,B+\text{coſ.}\,A}{\text{Coſ.}\,B-\text{coſ.}\,A} = \frac{\text{cot.}\,\overline{\tfrac{1}{2}A+\tfrac{1}{2}B}}{\text{tang.}\,\overline{\tfrac{1}{2}A-\tfrac{1}{2}B}}$ (d°) $= \frac{\text{ſec.}\,A+\text{ſec.}\,B}{\text{ſec.}\,A-\text{ſec.}\,B}$.

101. $\text{Sin.}\,A \times \text{ſin.}\,B = \tfrac{1}{2}\,\text{coſ.}\,\overline{A-B} - \tfrac{1}{2}\,\text{coſ.}\,\overline{A+B}$ (art. 26).

102. $\textit{Sin.}\ A \times \textit{coſ.}\ B = \frac{1}{2}\textit{ſin.}\ \overline{A+B} + \frac{1}{2}\textit{ſin.}\ \overline{A-B}$ (*art.* 26).

103. $\textit{Coſ.}\ A \times \textit{ſin.}\ B = \frac{1}{2}\textit{ſin.}\ \overline{A+B} - \frac{1}{2}\textit{ſin.}\ \overline{A-B}$ (d°).

104. $\textit{Coſ.}\ A \times \textit{coſ.}\ B = \frac{1}{2}\textit{coſ.}\ \overline{A+B} + \frac{1}{2}\textit{coſ.}\ \overline{A-B}$ (d°).

105. $\textit{Tang.}\ \overline{A+B} = \frac{\overline{\textit{tang.}\ A + \textit{tang.}\ B} \times R^2}{R^2 - \textit{tang.}\ A \times \textit{tang.}\ B}$ (*art.* 44).

106. $\textit{Tang.}\ \overline{A-B} = \frac{\overline{\textit{tang.}\ A - \textit{tang.}\ B} \times R^2}{R^2 + \textit{tang.}\ A \times \textit{tang.}\ B}$ (d°).

107. $\textit{Tang.}\ \overline{A+45^\circ} = \frac{\overline{R + \textit{tang.}\ A} \times R}{R - \textit{tang.}\ A} = \frac{R^2}{\textit{tang.}\ \overline{45^\circ - A}}$ (*art.* 46).

108. If A exceeds 45°. $\frac{\textit{Tang.}\ A + R}{\textit{Tang.}\ A - R} = \frac{R^2}{\textit{tang.}\ \overline{A - 45^\circ}}$ (d°).

CHAP. II.

Containing the general Properties of right or oblique-angled ſpherical Triangles, and their Reſolution by analogies.

SECTION I.

Of ſpherical Triangles in general.

DEFINITIONS.

109. ANY portion of a ſpherical ſurface, bounded by three arcs of great circles, is called a *ſpherical Triangle.* Fig. 9.

COROLLARY.

110. Hence it follows, that the ſmall circles of the ſphere do not fall under the conſideration of ſpherical Trigonometry, ſince ſuch only are uſed therein as have the ſame centre with the ſphere itſelf.

111. Every ſpherical, as well as plane triangle hath eſſentially three ſides and three angles: and if any three of theſe ſix parts be given, by the rules of Trigonometry, the reſt may be found.

SCHOLIUM.

112. In plane Trigonometry, the knowledge of the three angles is not ſufficient for obtaining the three ſides; for, in this caſe, the relations only of the three ſides can be had, and not their abſolute values; whereas, in ſpherical Trigonometry, where the ſides are circular arcs, whoſe values depend on the number of degrees they contain, when the three angles are given, the ſides will alſo become

known.——But there is yet another more remarkable difference between plane and ſpherical Trigonometry; which is, that in the former, two angles always determine the third; whereas in the latter they never do: and therefore it follows, that the definition as above ſtated is in ſtrictneſs applicable only to ſpherical Trigonometry; as will more clearly appear from the ſequel.

113. The ſides of a ſpherical triangle are all arcs of great circles, which, by their interſection upon the ſurface of the ſphere, conſtitute the ſaid triangle.

114. The angle, which is contained between the arcs of two great circles cutting each other upon the ſurface of the ſphere, (and already defined to be the ſides of a ſpherical triangle), is called a ſpherical angle; the meaſure whereof is known from the Elements of Geometry to be equal to that, which is formed by two lines iſſuing from the ſame point of, and perpendicular to, the common ſection of the planes which determine the containing ſides.

COROLLARY.

Fig. 9&10 115. Hence it follows, that the ſurface of a ſpherical triangle, BAC, and the three planes which determine it, form a kind of triangular Pyramid, BGCA, whereof the vertex, G, is at the centre of the ſphere, the baſe a portion of the ſpherical ſurface, whilſt its faces, AGC, AGB, and BGC, are parts of great circles or circular ſectors, and at the ſame time form the ſides of the triangle, BAC.

116. A line, as P G p, perpendicular to the plane of a great circle, paſſing through the centre of the ſphere, and terminated by two diametrically oppoſite points at its ſurface, is called the *axis* of ſuch circle; and the points, P, p, where the *axis* meets the ſurface, are called the *poles* thereof.

of. Moreover, if we conceive an infinite number of leſſer circles, parallel to the ſaid great circle, this *axis* will be perpendicular to them likewiſe, and the points P and *p* be their poles alſo.

COROLLARY I.

117. Hence, each pole of a great circle is 90° diſtant from every point in its circumference, and all the arcs drawn from the pole of a little circle to its circumference are equal to each other.

COROLLARY II.

118. It likewiſe follows from hence, that all the arcs of great circles drawn through the poles of another great circle are perpendicular thereto; for, ſince they are great circles by the ſuppoſition, they all paſs through the centre of the ſphere, and conſequently through the axis of the ſaid circle. The ſame thing may be affirmed with reſpect to ſmall circles.

COROLLARY III.

119. Therefore, in order to find the *poles* of any circle, we need only deſcribe upon the ſurface of the ſphere two great circles perpendicular to its plane; and the points, where theſe circles interſect each other, will be the poles required.

COROLLARY IV.

120. It moreover follows from hence, that if from any point taken upon the ſurface of the ſphere we would draw an arch of a circle, which may meaſure the ſhorteſt diſtance from this point to the circumference of any given circle, we muſt ſo deſcribe this arch, that its prolongation may paſs

through the poles of the ſaid circle: and contrarily, if an arc paſs through the poles of a given circle, it will meaſure the ſhorteſt diſtance from any aſſumed point to the circumference thereof.

COROLLARY V.

Fig. 10. 121. If upon the ſides, AC and BC, of a ſpherical triangle, BCA, we take the arcs CL and CK, each 90°, and through the radii, GL and GK, draw the circular plane LGK, it is manifeſt, that the point C will be the pole thereof; and as the lines, GK, GL, are both perpendicular to the common ſection of the planes, AGC and BGC, they meaſure by their inclination the angle of theſe planes, and of conſequence the ſpherical angle BCA likewiſe.

COROLLARY VI.

122. It is alſo manifeſt, that every arc of a leſſer circle deſcribed from the pole C as centre, and containing the ſame number of degrees with the arc KL, is equally proper for meaſuring the angle ACB; though only arcs of great circles are uſed for this purpoſe.

COROLLARY VII.

123. Therefore, if a ſpherical angle be right, the arcs of the great circles which form it, paſs mutually through the poles of each other; and if the planes of two great circles contain their reſpective axes, or paſs through the poles of each other, the angle which they comprehend is a right one.

THEOREM I.

124. *Any two ſides of a ſpherical triangle,* BAC, *are greater than the third.*

DE-

DEMONSTRATION.

This propoſition is a neceſſary conſequence of this; that the ſhorteſt diſtance between any two points, taken upon the ſurface of the ſphere, is the arc of a great circle paſſing through theſe points. Q. E. D.

THEOREM II.

125. *The ſum of the three ſides of any ſpherical triangle is leſs than 360°.*

DEMONSTRATION.

Let the ſides, AC and BC, containing any angle A be produced till they meet again in D, and the arcs, DAC, DBC, will be each 180°, ſince all great circles cut each other into two equal parts: we ſhall therefore have, DAC+DBC=360°: but, by the laſt Theorem, DA and DB are greater than AB, and conſequently the three ſides, AB, AC, and BC, when taken together, leſs than 360°. Q. E. D. *Fig.* 10.

THEOREM III.

126. *The ſum of the three angles of any ſpherical triangle is greater than two right angles, and leſs than ſix.*

DEMONSTRATION.

If the ſides, AB, AC, and BC, of the ſpherical triangle, ABC, be ſuppoſed indefinitely ſmall, the interſections, formed between the planes which determine theſe ſides, will approach indefinitely near to right lines, and the ſpherical ſurface indefinitely near to a plane ſurface: therefore, the ſaid triangle may in this caſe be conſidered as a plane triangle; but it is well known, that the angles of ſuch a triangle are equal to two right ones; and conſequently, whilſt the ſides of the ſpherical triangle, ABC, are of a finite magnitude, the ſum of its angles will be always greater than two right angles. Q. E. 1°. D.

2. It will readily appear from an inſpection of the 10th *fig.* that each angle of the ſpherical triangle ABC may be obtuſe; but at the ſame time ſuch, that the arc which is the meaſure thereof may be leſs than 180°; ſince then the angle vaniſhes: therefore, if we ſuppoſe all the angles of the ſaid triangle obtuſe, their ſum cannot poſſibly be ever equal to ſix right ones. Q. E. 2°. D.

COROLLARY.

127. Hence it follows, that a ſpherical triangle may have its three angles either right or obtuſe; and therefore, the knowledge of any two angles is not ſufficient for diſcovering the value of the third.

128. If the three angles of a ſpherical triangle be right or obtuſe, the three ſides are likewiſe equal to, or greater than, 90°; and if each of the angles be acute, each of the ſides is alſo leſs than 90°; and contrarily.

SCHOLIUM.

129. From theſe Theorems we may perceive, what the difference between plane and ſpherical triangles in reality is. But if, in ſome caſes, their difference is ſo great; yet, on the other hand, there are ſeveral properties, which are common to ſpherical with plane triangles, and which are demonſtrated exactly in the ſame manner. Thus, for inſtance, it might be here proved,

(as well as in elementary Geometry), that two ſpherical triangles are equal to each other: 1°, when the three ſides of the one are equal to the three ſides of the other; 2°, when they have an equal angle contained between equal ſides, and 3°, when they have equal angles above equal baſes. We might likewiſe ſhew, that a ſpherical triangle is equilateral, iſoſceles or ſcalenous, according as it hath three equal, two equal, or three unequal, angles; and contrarily; and laſtly, that the greateſt ſide is always oppoſite to the greateſt angle, and the leaſt ſide to the leaſt angle. The Demonſtrations of all theſe truths are exactly the ſame with thoſe in the correſponding caſes of plane triangles, ſo that it will be judged unneceſſary to ſpecify them in this place.

THEOREM IV.

130. *If, from the three angles,* B, A *and* C, *of a ſpherical triangle,* BAC, *as poles, there be deſcribed upon the ſurface of the ſphere three arcs of great circles,* DF, DE *and* FE, *which, by their interſections, form another ſpherical triangle,* DEF; *each ſide of this new triangle will be the ſupplement to the angle which is at its pole, and each of its angles the ſupplement to that ſide in the triangle* BAC, *whereto it is oppoſite.* *Fig.* 11

DEMONSTRATION.

Let the ſides, AB, AC and BC, of the triangle BAC, be produced till they meet thoſe of the triangle DEF in the points, I, L; M, N; G, K: then, ſince the point A is the pole of the arc DILE, the diſtance of the points A and E will be 90°; and, ſince C is the pole of the arc EF, the points C and E will likewiſe be 90° diſtant; therefore, by *art.* 117, the point E is the pole of the arc AC. We might prove, in like manner, that F is the pole of BC, and D that of the arc AB.——This conſtruction

being

being well underſtood, we ſhall have, DL=90°, and IE=90°; and therefore, DL+IE, or DI+IL+EL +IL, or, DE+IL=180°: conſequently, the arc DE is the ſupplement to the angle BAC, meaſured by the arc IL (*art.* 121). We might prove, in the ſame manner, that EF is the ſupplement to the angle BCA, meaſured by the arc MN, and that DF is the ſupplement to the angle ABC, meaſured by GK: whence it follows, that each ſide of the triangle DEF is the ſupplement to the angle in the triangle BAC, which is at its pole. Q. E. 1°. D.

2°. The angles of the ſame triangle are ſupplemental to the ſides of the triangle ABC; for, ſince the arcs AL and BG are each 90°, we ſhall have, AL+BG, or, GL+AB=180°; but GL is the meaſure of the angle EDF by *art.* 121, and conſequently AB the ſupplement thereof. We might prove in the ſame manner, that AC and BC are the ſupplements to the angles at E and F: wherefore all the angles of the triangle DEF are ſupplemental to the ſides oppoſite to them in the triangle BAC. Q. E. 2°. D.

SECT. II.

Of the Reſolution of right-angled Spherical Triangles.

Preparation to the following Theorems.

Fig. 12. 131. Let GBPQ be a pyramid, compoſed of four right-angled triangles, GBQ, GBP, GPQ and BPQ; and let, AB, AC and BC, be three circular arcs, deſcribed from the centre G with the radius GB, and evidently forming a ſpherical triangle, BAC, right-angled at A; ſince the planes, GQP, GBP, are perpendicular to each other: then, if we make the radius equal to unity, we ſhall eaſily obtain the values ſpecified in the following Table.

Table, for all the parts of the ſaid triangle; by recollecting that tang. = [illegible], and cot. = $\frac{1}{\text{tang.}}$

	for its [illegible]	for its Coſ.	for its tang.
The arc BC, or ang. QGB	$\frac{BQ}{GQ}$	$\frac{BG}{GQ}$	$\frac{BQ}{BG}$
The arc BA, or ang. BGP	$\frac{BP}{GP}$	$\frac{GB}{GP}$	$\frac{BP}{BG}$
The arc AC, or ang. PGQ	[illegible]	$\frac{GP}{GQ}$	[illegible]
The ang. ABC, or QBP	$\frac{PQ}{BQ}$	$\frac{BP}{BQ}$	[illegible]
The ang. BCA	$\frac{BP\times GQ}{BQ\times GP}$	$\frac{QP\times BG}{GP\times BQ}$	[illegible]

The expreſſions in cotangents being obtained by only inverting thoſe of the tangents, are here omitted.

In

132. In order to ſhew how theſe ſeveral expreſſions are demonſtrated, it will be ſufficient to give the demonſtration of the firſt line only. And to do this, I aſſume any line, GR, which I regard as the ſine total of the Tables; then I let fall RS perpendicularly upon GB, and, it is evident that, this line will be the *ſine* of the arc BC, and GS the *coſine* thereof. This being premiſed, the ſimilar triangles, GRS, GQB, give the two following proportions; QG : QB :: GR (1) : RS = *ſin.* BC $= \frac{QB}{QG}$, and, QG : GB :: GR (1) : GS = *coſ.* BC $= \frac{GB}{QG}$; from whence I immediately deduce, *tang.* BC $= \frac{BQ}{BG}$, and *cot.* BC $= \frac{BG}{BQ}$: the other expreſſions are demonſtrated exactly in the ſame manner. To prove the truth of the following Theorems, the particular values of the terms of the proportions to be demonſtrated need only be ſubſtituted, and there will be always found a perfect equality ſubſiſting between the product of the extremes and that of the means.

THEOREM I.

Fig. 13. 133. *In any right-angled ſpherical triangle, BAC, the ſine total is to the ſine of the hypothenuſe, as the ſine of either angle is to that of its oppoſite leg, and converſely.*

N. B. By this Theorem, the expreſſions of the *ſine* and *coſine* of the angle BCA were obtained; as well as thoſe of its *tangent* and *cotangent*.

THEOREM II.

134. *In any right-angled ſpherical triangle we ſhall always have, as radius is to the coſine of either angle, ſo*

ſo is the tangent of the hypothenuſe to the tangent of the leg adjacent to this angle; that is, R : *coſ.* B : *tang.* BC : : *tang.* AB; or, R : *coſ.* C : : *tang.* BC : *tang.* AC.

THEOREM III.

135. *We ſhall likewiſe have in every right-angled ſpherical triangle; as the ſine total is to the coſine of one of the legs, ſo is the coſine of the other leg to that of the hypothenuſe*; or, which is the ſame, R : *coſ.* AB : : *coſ.* AC : *coſ.* BC.

THEOREM IV.

136. *We ſhall moreover have; as radius is to the ſine of either angle, ſo is the coſine of the adjacent leg to the coſine of the other angle*; or, R : *ſin.* B or *ſin.* C : : *coſ.* AB or *coſ.* AC : *coſ.* C or *coſ.* B.

THEOREM V.

137. *As radius is to the ſine of one of the legs, ſo is the tangent of its adjacent angle to the tangent of the other leg*; or, R : *ſin.* AB : : *tang.* B : *tang.* AC, and, R : *ſin.* AC : : *tang.* C : *tang.* AB.

THEOREM VI.

138. *The radius is to the cotangent of one of the angles, as the cotangent of the other angle is to the coſine of the hypothenuſe; or* (which comes to the ſame) *the radius is to the coſine of the hypothenuſe, as the tangent of one angle is to the cotangent of the other angle*; that is, R : *cot.* B : : *cot.* C : *coſ.* BC; or, R : *coſ.* BC : : *tang.* B : *cot.* C : : *tang.* C : *cot.* B.

COROLLARY I.

139. From the ſecond Theorem it follows that, if two right-angled ſpherical triangles have one leg

common; the *tangents* of their hypothenuſes are in the inverſe ratio of the *coſines* of the angles adjacent to this leg.

COROLLARY II.

140. From the third Theorem it likewiſe follows that, if two right-angled ſpherical triangles have one leg common, the *coſines* of their hypothenuſes are as the *coſines* of their other legs.

COROLLARY III.

141. It alſo follows from the fourth Theorem that, if two right-angled ſpherical triangles have one leg common, the *coſines* of the angles oppoſite to this leg are to each other as the *ſines* of the adjacent angles.

COROLLARY IV.

142. It moreover follows from the fifth Theorem that, if two right-angled ſpherical triangles have one leg common, the *ſines* of their other legs are reciprocally as the *tangents* of the angles adjacent to their legs.

COROLLARY V.

143. Laſtly, if two right-angled ſpherical triangles have one angle common, the *ſines* of their hypothenuſes are as the *ſines* of their legs oppoſite to this angle, and the *tangents* of theſe legs as the *ſines* of the legs adjacent to the ſaid angle.—— The firſt of theſe truths is a direct conſequence of the firſt Theorem, and the ſecond of the fifth.†

SCHOLIUM.

144. The ſix preceding Theorems contain whatever is neceſſary for obtaining the ſolutions of

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† See Emerſon's Elements of Trigonometry, 2d edition.

all the poſſible caſes of right-angled ſpherical triangles; as we may be eaſily convinced by Fig. P. at the end of this Chapter. However, as there may be ſome difficulty in retaining, as well as ſome danger in confounding them, we ſhall alſo add the famous Theorem of *Neper*, whereby they are all reduced to two *general* caſes, that may be very eaſily remembered, provided we perfectly underſtand the following definitions; which are abſolutely neceſſary to convey a juſt idea of its nature.

DEFINITIONS.

145. When three parts of a right-angled ſpherical triangle are ſo ſituated, that two of them immediately touch the third, or are ſeparated from it by the right-angle only; theſe two parts are ſaid to be *adjacent* to the third, which is called the *mean* or *middle* part.

146. But when three of the parts of a right-angled ſpherical triangle are ſo ſituated, that between one of theſe parts, conſidered as the *mean*, and each of the other two, there is always found ſome other part of the ſame triangle; then are theſe two parts ſaid to be *ſeparated*.——The right-angle is not ſuppoſed to ſeparate its contiguous parts.—— This being premiſed;

If the *middle* parts be	AB	the *adjacent* ones, or *extremes conjunct*, will be	AC & B	and the *ſeparated* ones, or *extremes disjunct*,	BC & C.
	AC		AB & C		BC & B.
	BC		B & C		AC & AB.
	B		AB & BC		AC & C.
	C		AC & BC		AB & B.

GENERAL THEOREM.

147. *If, in any right-angled ſpherical triangle, the complements of the ſides containing the right-angle be ſubſtituted for the ſides themſelves, we ſhall always have,*

the rectangle under the ſine total and coſine of the middle part, equal to the rectangle under the cotangents of the adjacent *parts, or to that under the ſines of the* ſeparated *parts.*

DEMONSTRATION.

We have taken care to particulariſe in the Table for the ſolutions of right-angled triangles all the caſes of this Theorem, denoting by A, the caſes which have relation to the *adjacent* parts, and by S, thoſe of the *ſeparated* parts; ſo that the truth of the general Theorem is demonſtrated by that of the particular Theorems correſponding to thoſe caſes. However, it may likewiſe be proved from the Table in *art.* 131, by ſubſtituting for each particular term the values therein ſpecified: thus, for inſtance, if we would demonſtrate the firſt line in *art.* 146, we need only prove that, R×*ſin.* AB = *tang.* AC×*cot.* B = *ſin.* BC×*ſin.* C. Now, by taking the expreſſions of theſe lines, as given in *art.* 131, we ſhall get, $R\times\frac{BP}{GP}=\frac{QP\times BP}{GP\times QP}=\frac{BQ\times BP\times GQ}{GQ\times BQ\times GP}$; all which become manifeſtly the ſame, after expunging the terms which deſtroy in the two laſt fractions.——It would be the ſame with reſpect to the other lines; ſo that we may very juſtly conclude, that our Theorem is true in all poſſible caſes. Q. E. D.

Of the values of the Angles of a right-angled ſpherical Triangle, with regard to the Sides which contain the right angle, &c.

THEOREM.

148. *In any right-angled ſpherical triangle,* BAC, or *b*AC, *the angles above the hypothenuſe are always*

of the ſame affection with their oppoſite ſides. 2°. the hypothenuſe is leſs or greater than a quadrant, according as the legs containing the right-angle are of the ſame or different affection; that is to ſay, *according as they are both acute or obtuſe, or the one acute and the other obtuſe.*

DEMONSTRATION.

If the ſides AB and AC be each 90°; then, ſince the angle A is right by ſuppoſition, the points B and C will become the poles of the arcs AC and AB; and conſequently, the angles B and C be both right, (as being meaſured by arcs which are ſuppoſed 90°); that is to ſay, of the ſame affection with their oppoſite ſides—Moreover, if the legs AB and AC be both acute, the angles which are oppoſite to them will be ſo likewiſe; as may be eaſily proved thus. Let the ſide AC be produced to F, ſo that AF may be 90°; then, as the point F will, by *art.* 117, be the pole of the arc AB, the angle B will be alſo 90°; and conſequently, the angle ABC, which is leſs than the angle ABF, neceſſarily acute. We might prove, in like manner, that the angle at C is acute, when the ſide AB, which is oppoſite to it, is acute.——It is equally manifeſt, that the angles B and C, in the triangle B*a*C, right-angled at *a*, are obtuſe, when the ſides, *a*B, *a*C, which are oppoſite to them, are obtuſe. ——If one of the ſides A*b* is obtuſe, and the other ſide AC acute, as in the right-angled triangle *b*AC, the angle at C will be likewiſe obtuſe, and that at *b* acute. For, having taken upon AB*b* the arc AG 90°, and drawn from the point G, to the point C, the arc GC, the angle ACG will be right, by *art.* 118, ſince G is the pole of the arc AC; from whence it neceſſarily follows, that the angle AC*b* will be obtuſe. For the ſame reaſon, the angle *ab*C, in the right-angled triangle *ba*C, will be obtuſe; ſince it is oppoſite to the obtuſe ſide *a*C; and conſequently, the ſupplement thereof A*b*C acute; that is, of the ſame affection with its oppoſite ſide. Therefore, in general, the angles above the hypothenuſe are of the ſame affection with the ſides which are oppoſite to them. Q. E. 1°. D.

2°. It is evident, that BC, whether conſidered as the hypothenuſe of the triangle BAC, or that of the triangle B*a*C, is leſs than BF; and that the hypothenuſe *b*C, in the triangle *ba*C, is greater than *b*F: from whence it follows, that the hypothenuſe of any right-angled ſpherical triangle is always leſs than 90°, when the two legs are of the ſame affection; and greater, when they are of different. Q. E. 2°. D.

It will eaſily appear, that the converſe of this Theorem is true in all its parts, viz. that, if the angles above the hypothenuſe are of the ſame or different affection, their oppoſite ſides will be ſo likewiſe; and that the legs are of the ſame or different affection, according as the hypothenuſe is leſs or greater than a quadrant; but one or both of them 90°, when it is exactly a quadrant.

PROBLEM I.

Fig. 13. 149. *The hypothenuſe*, BC, *of a right-angled ſpherical triangle*, BAC, *together with the ſum or difference of the two legs*, AB *and* AC, *being given; to determine the triangle.*

SOLUTION.

Since we have, by *art.* 135, R : *coſ.* AB : : *coſ.* AC : *coſ.* BC, we ſhall alſo have, *coſ.* AB × *coſ.* AC = R × *coſ.* BC. But, by *art.* 104, *coſ.* AB × *coſ.* AC

=½

$= \frac{1}{2}\,\textit{coſ.}\,\overline{AB+AC} + \frac{1}{2}\,\textit{coſ.}\,\overline{AB-AC}$; and conſequently, $2R \times \textit{coſ.}\,BC - \textit{coſ.}\,\overline{AB+AC} = \textit{coſ.}\,\overline{AB-AC}$, or, $2R \times \textit{coſ.}\,BC - \textit{coſ.}\,\overline{AB-AC} = \textit{coſ.}\,\overline{AB+AC}$. Q. E. I.

COROLLARY.

150. Hence, if the legs be equal, we ſhall have, $2\,\textit{coſ.}\,BC - R = \textit{coſ.}\,2AB$* or $\textit{coſ.}\,2AC$; which ſhews us, that in this caſe, the *coſine* of the double of either of the legs is equal to twice the *coſine* of the hypothenuſe leſs the ſine total.

PROBLEM. II.

151. *Given one of the legs, and the ſum or difference of the hypothenuſe and other leg; to find the hypothenuſe.*

SOLUTION.

Since we have, by *art.* 135, $R : \textit{coſ.}\,AB :: \textit{coſ.}\,AC : \textit{coſ.}\,BC$, we ſhall likewiſe have, *componendo et dividendo*, $R + \textit{coſ.}\,AB : R - \textit{coſ.}\,AB :: \textit{coſ.}\,AC + \textit{coſ.}\,BC : \textit{coſ.}\,AC - \textit{coſ.}\,BC$; which, by *art.* 76 and 100, becomes, $\textit{cot.}\,\frac{AB}{2} : \textit{tang.}\,\frac{AB}{2} :: \textit{cot.}\,\frac{BC+AC}{2} : \textit{tang.}\,\frac{BC-AC}{2}$; from whence it follows, that, if we know the ſum or difference of the hypothenuſe and one leg, we ſhall be able by this analogy to find the difference or ſum of the hypothenuſe and other leg.

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* Becauſe, $\textit{coſ.}\,AB \times \textit{coſ.}\,AC = R \times \textit{coſ.}\,BC$, then becomes, $\textit{coſ.}^2\,AB$ or $\textit{coſ.}^2\,AC = R \times \textit{coſ.}\,BC$: but, by *art.* 32, $\textit{coſ.}^2\,AB$ or $\textit{coſ.}^2\,AC = \frac{R^2 + R \times \textit{coſ.}\,2AB}{2}$ or $\frac{R^2 + R \times \textit{coſ.}\,2AC}{2}$; and therefore, $2\,\textit{coſ.}\,BC - R = \textit{coſ.}\,2AB$ or $\textit{coſ.}\,2AC$.

PROBLEM III.

152. *One of the angles above the hypothenuſe, with the ſum or difference of the hypothenuſe and adjacent leg in a right-angled ſpherical triangle, being given; to determine the triangle.*

SOLUTION.

Since we have, by *art.* 134, R : *coſ.* B : : *tang.* BC : *tang.* AB, we ſhall alſo have, *componendo et dividendo*, R + *coſ.* B : R − *coſ.* B : : *tang.* BC + *tang.* AB : *tang.* BC − *tang.* AB; which, by *art.* 76 and 87, become, *cot.*^{2} $\frac{1}{2}$ B : R^2, or, *cot.* $\frac{1}{2}$ B : *tang.* $\frac{1}{2}$ B : : *ſin.* $\overline{BC+AB}$: *ſin.* $\overline{BC-AB}$. Q. E. I.

PROBLEM IV.

153. *Given the hypothenuſe, and the ſum or difference of the angles above the hypothenuſe; to find theſe angles.*

SOLUTION.

By *art.* 138, we have, R : *coſ.* BC : : *tang.* C : *cot.* B; which, by proceeding as in the laſt Problem, gives, R^2 : *tang.*^{2} $\frac{1}{2}$ BC : : *coſ.* $\overline{B-C}$: *coſ.* $\overline{B+C}$;* from whence the ſolution required is eaſily obtained.——But, if the two angles be equal, we ſhall have, R^2 : *tang.* $\frac{1}{2}$ BC : : *tang.* $\frac{1}{2}$ BC : *coſ.* 2 B or *coſ.* 2 C. Q. E. I.

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* Becauſe, R : *coſ.* BC : : *tang.* C : *cot.* B : : *tang.* C : *tang.* $\overline{90^\circ-B}$; we ſhall likewiſe have, R + *coſ.* BC : R − *coſ.* BC : : *tang.* C + *tang.* $\overline{90^\circ-B}$: *tang.* C − *tang.* $\overline{90^\circ-B}$; and conſequently, by *art.* 76 and 87, R^2 : *tang.*^{2} $\frac{1}{2}$BC : : *ſin.* $\overline{C+90^\circ-B}$: *ſin.* $\overline{C-90^\circ+B}$: : *coſ.* $\overline{B-C}$: *coſ.* $\overline{B+C}$.

PROBLEM V.

154. *Given in two right-angled ſpherical triangles,* BAC, BDG, *which have one angle common, the legs,* AC *and* DG, *oppoſite to this angle, with the ſum or difference of their hypothenuſes; to determine theſe triangles.* Fig. 14.

SOLUTION.

Since, by *art.* 143, *ſin.* DG : *ſin* AC : : *ſin.* BG : *ſin.* BC, we ſhall likewiſe have, *componendo et dividendo*, $\text{ſin. DG}+\text{ſin. AC} : \text{ſin. DG}-\text{ſin. AC} :: \text{ſin. BG}+\text{ſin. BC} : \text{ſin. BG}-\text{ſin. BC}$, and conſequently, by *art.* 95, $\text{tang.}\,\frac{DG+AC}{2} : \text{tang.}\,\frac{DG-AC}{2} :: \text{tang.}\,\frac{BG+BC}{2} : \text{tang.}\,\frac{BG-BC}{2}$; from whence the ſolution of the Problem will be very eaſy, ſince three terms in this proportion will always be known. Q. E. I.

PROBLEM VI.

155. *Let there be ſtill two right-angled ſpherical triangles, which have one angle common, and let the legs oppoſite to this angle with the ſum or difference of the adjacent legs be given; to determine the triangles.*

SOLUTION.

Since we have, by *art.* 143, *tang.* DG : *tang.* AC : : *ſin.* BD : *ſin.* BA, we ſhall likewiſe have, $\text{tang. DG}+\text{tang. AC} : \text{tang. DG}-\text{tang. AC} :: \text{ſin. BD}+\text{ſin. BA} : \text{ſin. BD}-\text{ſin. BA}$; from whence we deduce, by making ſubſtitutions ſimilar to the preceding, $\text{ſin.}\,\overline{DG+AC} : \text{ſin.}\,\overline{DG-AC} :: \text{tang.}\,\frac{BD+BA}{2} : \text{tang.}\,\frac{BD-BA}{2}$†. Q. E. I.

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† See *Simpſon's* Trigonometry, p 69, &c.

SECT. III.

Containing the Reſolution of oblique-angled ſpherical Triangles.

THEOREM I.

Fig. 15. 156. *In any ſpherical triangle,* BAC, *the ſines of the angles are to each other directly as the ſines of their oppoſite ſides.*

DEMONSTRATION.

From any angle, A, of the ſpherical triangle BAC, let fall the arc AD perpendicularly upon the baſe BC: then, in the right-angled ſpherical triangles, BAD, CAD, we ſhall have, by *art.* 133, R : *ſin.* AB : : *ſin.* B : *ſin.* AD, and, R : *ſin.* AC : : *ſin.* C : *ſin.* AD; whence we immediately deduce, *ſin.* AB×*ſin.* B=*ſin.* AC×*ſin.* C; and conſequently this proportion; *Sin.* B : *ſin.* C : : *ſin.* AC : *ſin.* AB. Q. E. D.

DEFINITIONS.

157. The angles, BAD, CAD, which the ſides, AB, AC, containing the angle BAC, form with the perpendicular, are called the *ſegments of the vertical angle*; whether the perpendicular falls within or without the triangle BAC.

2°. In like manner, the parts, BD, DC, of the ſide BC, contained between the points, B, C, and the point D, where the perpendicular AD meets this ſide, are called the *ſegments of the baſe*; whether the baſe be produced or not.

3°. With regard to the ſegments, BAD, CAD, of the angle BAC, the containing ſides, AB, AC, will be called *adjacent parts*, as being ſo in effect; but the angles, B, C, above the baſe BC, *ſeparated parts*,

parts, ſince between theſe angles and the ſaid ſegments are found the ſides, BA, CA.

4°. With reſpect to the ſegments, BD, CD, of the baſe, the angles, B, C, will be *adjacent parts*, and the ſides, BA, CA, *ſeparated parts.*—— This being premiſed, it will be eaſy to remember the two parts of the following Theorem.

THEOREM II.

158. *If from any angle*, A, *of a ſpherical triangle*, BAC, *we let fall a perpendicular*, AD, *upon the oppoſite ſide*, BC, *(produced if neceſſary); we ſhall always have;*

1°. The ſines of the ſegments of this angle as the coſines of the ſeparated parts; and the coſines thereof as the cotangents of the adjacent parts.

2°. The ſines of the ſegments of the baſe as the cotangents of the adjacent parts; and the coſines thereof as the coſines of the ſeparated parts:

that is
1°. *Sin.* BAD : *ſin.* CAD : : *coſ.* B : *coſ.* C.
2°. *Coſ.* BAD : *coſ.* CAD : : *cot.* AB : *cot.* AC.
3°. *Sin.* BD : *ſin.* CD : : *cot.* B : *cot.* C.
4°. *Coſ.* BD : *coſ.* CD : : *coſ.* AB : *coſ.* AC.

DEMONSTRATION.

In the right-angled ſpherical triangle BDA we ſhall have, by *Theorem* IV. R : *coſ.* AD : : *ſin.* BAD : *coſ.* B; and likewiſe, in the right-angled triangle CDA, by the ſame, R : *coſ.* AD : : *ſin.* CAD : *coſ.* C; therefore, *Sin.* BAD : *ſin.* CAD : : *coſ.* B : *coſ.* C. Q. E. 1°. D.

2°. The ſame triangles will alſo give, by *Theorem* II. R : *coſ.* BAD : : *cot.* AD : *cot.* AB, and, R : *coſ.* CAD : : *cot.* AD : *cot.* AC; therefore, ſince theſe two proportions have the ſame antecedents, we ſhall have, *Coſ.* BAD : *coſ.* CAD : : *cot.* AB : *cot.* AC. Q. E. 2°. D.

3°. The ſame triangles will moreover give, by *Theorem* V. the two following proportions ; R : *ſin.* BD : : *cot.* AD : *cot.* B, and, R : *ſin.* CD : : *cot.* AD : *cot.* C ; therefore, ſince the antecedents are equal, the conſequents will be likewiſe proportional, and give, *Sin.* BD : *ſin.* CD : : *cot.* B : *cot.* C. Q. E. 3°. D.

Laſtly, by *Theorem* III. we ſhall have in the triangle BAD, R : *coſ.* BD : : *coſ.* AD : *coſ.* AB ; and in the triangle CDA, R : *coſ.* DC : : *coſ.* AD : *coſ* AC ; from whence it follows, that *Coſ.* BD : *coſ.* CD : : *coſ.* AB : *coſ.* AC. Q. E. 4°. D.

SCHOLIUM.

159. The four parts of this Theorem are, upon the whole, of equal import with the Corollaries, which we have ſubjoined to the ſix Theorems upon right-angled triangles ; only, we have here attempted to render them ſomewhat eaſier to be remembered, by means of the definitions which we have prefixed to this Theorem.

Preparation to the following Theorems.

PROBLEM.

Fig. 16. 160. *Any ſpherical triangle,* BAC, *being given, one ſide whereof,* AB, *is ſuppoſed to be upon the circumference of a great circle,* ABRF*ar* ; *it is required to find the orthographic projection of this triangle upon the plane of the ſaid circle* ; that is to ſay, *that which is formed by lines, let fall from all the points of the ſides of the triangle,* A B C, *perpendicularly upon the plane,* ABRF*ar.*

SOLUTION.

From the extremities, A, B, of the arc AB, let there be drawn to the centre G the radii, GA, GB; alſo, through this centre (which is likewiſe that of the ſphere), let a plane or great circle, *r*DR*o*, be conceived to paſs perpendicularly to the plane AR*ar*, ſo that their common ſection, R*r*, may be perpendicular to the radius AG; laſtly, let the arc AC be produced till it meet the circumference *r*DR*o*, in the point D; from whence, let the line DG be drawn to the centre G, as alſo the line D*d* perpendicular to the diameter R*r*. This being done, it is evident, that the angle DGR is equal to the angle BAC, which is formed by the interſection of the planes, BAG, CAG, (ſince the lines, DG, RG, are both perpendicular to their common ſection, AG); and, that the angle DG*r* is equal to the ſupplement thereof. ——— It is equally manifeſt, that, if from the point C we let fall a line, C*c*, perpendicularly upon the plane AR*ar*, the point *c* will be the projection of the angular point C. Moreover, if through the point C the plane of a little circle, *l*CL, be made to paſs in a poſition parallel to the plane *r*DR*o*, the common ſection, *l*L, of this plane, and that of the great circle AR*ar*, will be alſo perpendicular to the radius AG, and determine each of the arcs, AL, A*l*, equal to the arc AC; and the projection of the point C will be found in a point of this line: but it will be likewiſe found (for the ſame reaſon) in a line, F*f*, which is the common ſection of the plane of the great circle AR*ar*, and that of the little circle *f*CF perpendicular thereto; provided this line be alſo perpendicular to the radius BG, and of courſe determine each of the arcs, BF, B*f*, equal to the arc BC: and therefore, when the three ſides of a triangle, BAC, are known, it is eaſy to perceive, from hence, how the projection of an angle C may be determined by the interſection of the lines, Ll, Ff, upon the plane of the circle ABR*ar*. Q. E. I. *et* D.

COROLLARY I.

161. The right-angled triangles, DdG, CcH, being ſimilar, on account of the parallel lines whereof they are compoſed, we ſhall have, $DG : CH$ or $rG : lH :: dG : cH$; that is to ſay, R : *ſin.* AC :: *coſ.* BAC : cH; therefore, as we ſhould have the ſame proportion for all the projected points of the arc AC, it follows, that the projection of this arc upon the plane of the circle ABR*ar* is an ellipſe,* whereof AG is half the tranſverſe axis, and dG half the conjugate.

COROLLARY II.

162. It likewiſe follows, from the preceding Corollary, that the *orthographic* projection of a circle, or part of a circle, is always an ellipſe, or part of an ellipſe; whereof half the tranſverſe axis is equal to the *ſine total*, and half the conjugate to the *coſine* of the angle contained between the plane of the ſaid circle and the plane of projection.

SCHOLIUM.

163. In the two following Chapters, we ſhall give a more minute Theory of the *orthographic* pro-

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* See *Robertſon*'s Tranſlation of *De la Caille*'s Aſtronomy, p. 160, &c.

jection, and of its application both to the graphical and analytical reſolution of all the caſes of ſpherical triangles. What we have ſaid thereon, in this place, muſt be conſidered only as a *Lemma* abſolutely neceſſary for the perfect underſtanding of the following Theorem; the difficulty whereof will be very eaſily ſurmounted, provided we have conceived a juſt idea of the nature of the planes, mentioned in the foregoing conſtruction.

THEOREM III.

264. *In any ſpherical triangle,* BAC, *this proportion will always be true: as the product of the ſines* Fig. 17. *of the ſides,* AB, AC, *containing any angle,* BAC, *is to the product of the ſines of the differences of theſe and half the ſum of the three ſides, ſo is radius ſquared to the ſquare of the ſine of half the ſaid angle;* or, which is the ſame, $\text{Sin.}\,AB\times\text{ſin.}\,AC : \text{ſin.}\,\frac{AB+AC+BC}{2}-AC\times\text{ſin.}\,\frac{AB+AC+BC}{2}-AB :: R^2 : \text{ſin.}^2\,\frac{1}{2}\,BAC$.

Conſtruction neceſſary to the Demonſtration.

Upon the plane of the circle ABR*ar*, and on each ſide of the point A, let there be taken the arcs, AL, A*l*, equal to the arc AC; likewiſe, on each ſide of the point B, let the arcs, BF, B*f*, be taken equal to the arc BC, and afterwards the chords, L*l*, F*f*, drawn reſpectively perpendicular to the radii, GA, GB: then (from what hath been demonſtrated in the preceding Problem) it is evident, that the interſection C of theſe chords will be the projection of the angle C, in the triangle BAC; and (from this conſtruction) that, BL=AC—AB; B*l*=AC+AB; LF=BF or BC—AC+AB; *lf*=AB+AC—BC, and, L*f*=BC+AC—AB. In the next place, let this proportion be made, H*l* : CH ::

Gr : Gd; or, which is the ſame, upon Gr, let there be taken, Gd, a fourth proportional to the three lines, Hl, CH, Gr, and this line will be the *coſine* of the angle BAC, as appears from *fig.* 16; and conſequently, if at the point *d* we raiſe a right line Dd, perpendicular to the radius Gr, it will determine the angle RGD equal to the angle BAC, and be the *ſine* thereof. Laſtly, from the point D, to the extremities of the diameter Rr, let there be drawn the chords, DR, Dr; upon which from the centre G let fall the perpendiculars, GS, Gs, and then from the points, S, s, the perpendiculars, SV, su, upon the diameter Rr; and it will manifeſtly appear, from this conſtruction, that RS is the *ſine* of half the angle BAC, and *rs* the *ſine* of half its ſupplement; that is, the *coſine* of half the ſaid angle.——So much being premiſed, we ſhall find no difficulty in the

DEMONSTRATION.

In the right-lined triangle CLF, the *ſines* of the angles being to each other as the halves of their oppoſite ſides, and the angle, LOF*, being manifeſtly equal to the angle AGB, its *ſine* will be equal to that of the arc AB, and we ſhall have this proportion; *ſin.* C or *ſin.* AB : *ſin.* F :: ½ LF : ½ CL; alſo, on account of the proportional lines, GR, HL, Gd, CH, we ſhall have, HL : GR :: ½ CL : VR = ½ Rd; and, likewiſe, ſince the lines, GR, RS, and RV, are in continued proportion, GR : VR :: $\overline{GR}^2 : \overline{RS}^2$: then, if we multiply the correſponding terms of theſe three proportions

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[*] Becauſe it is meaſured by half the ſum of the arcs, LF, *lf*; which, by what hath been ſaid above, are equal to 2AB.

together, and expunge ſuch quantities as are common both to the antecedents and conſequents, we ſhall get, *ſin.* AB×HL : *ſin.* F :: $\frac{1}{2}$ LF×$\overline{GR}^2$: $\overline{RS}^2$, or, *ſin.* AB×HL : *ſin.* F×$\frac{1}{2}$LF :: $\overline{GR}^2 : \overline{RS}^2$; but, HL = *ſin.* AC; *ſin.* F = *ſin.* $\frac{1}{2}$ LA*f* = *ſin.* $\frac{BC+AC-AB}{2}$ = *ſin.* $\overline{\frac{BC+AC+AB}{2}-AB}$, and, $\frac{1}{2}$ LF = *ſin.* $\frac{1}{2}$ LRF = *ſin.* $\frac{BC+AB-AC}{2}$ = *ſin.* $\overline{\frac{BC+AB+AC}{2}-AC}$; and therefore, if we ſubſtitute theſe values in the laſt proportion, it will become, *Sin.* AB × *ſin.* AC : *ſin.* $\overline{\frac{BC+AB+AC}{2}-AB}$ × *ſin.* $\overline{\frac{BC+AB+AC}{2}-AC}$: R^2 : *ſin.*^{2} $\frac{1}{2}$BAC. Q. E. D.

THEOREM IV.

165. *Let the ſame conſtruction be ſuppoſed as in the preceding Theorem, and I ſay that we ſhall likewiſe have this proportion; as the product of the ſines of the ſides,* AB, AC, *containing any angle,* BAC, *is to the product of the ſine of half the difference of theſe ſides and the third by the ſine of half the ſum of the three ſides, ſo is radius ſquared to the ſquare of the coſine of half the included angle*; that is, *Sin.* AB × *ſin.* AC : *ſin.* $\frac{AB+AC-BC}{2}$ × *ſin.* $\frac{AB+AC+BC}{2}$ :: R^2 : *coſ.*^{2} $\frac{1}{2}$ BAC.

DEMONSTRATION.

In the triangle C*lf*, the *ſines* of the angles are to each other as the halves of their oppoſite ſides: but, it is evident that, the angle at *f* is meaſured by half the arc FD*l*, which is the ſupplement to half the arc FAl, equal to half the ſum of the three ſides, AC, AB, BC: the arc lf is alſo manifeſtly equal to, AB+AC—BC; and therefore, the half of its chord will be the *ſine* of, $\frac{AB+AC-BC}{2}$. This being premiſed, we ſhall have, *ſin.* AB : *ſin.* f :: $\frac{1}{2} lf : \frac{1}{2} Cl$; by conſtruction, H$l$: Gr :: $\frac{1}{2}$ Cl : $ur = \frac{1}{2} dr$; and likewiſe, ſince the lines, Gr, rs, and ru, are in continued proportion, Gr : ur :: $\overline{Gr}^2 : \overline{rs}^2$; therefore, if we multiply the correſponding terms together, we ſhall get, *ſin.* AB×Hl : *ſin.* f :: $\frac{1}{2} lf \times \overline{Gr}^2 : \overline{rs}^2$, or, *ſin.* AB × H$l$: : $\frac{1}{2} lf \times$ *ſin.* f :: $\overline{Gr}^2 : \overline{rs}^2$; and, by ſubſtituting the value of each line, Sin. AB×*ſin.* AC : *ſin.* $\frac{AB+AC-BC}{2} \times$ *ſin.* $\frac{AB+AC+BC}{2}$ :: R² : *coſ.*² $\frac{1}{2}$ BAC. Q. E. D.

COROLLARY I.

166. Hence, if we call the ſum of the three ſides, s; the ſide oppoſite to the angle ſought, a; and the ſides including this angle, b, and c; we ſhall have; *ſin.* $\frac{1}{2}$ angle $= \frac{r\sqrt{\text{ſin.}\,\overline{\frac{1}{2}s-b}\times\text{ſin.}\,\overline{\frac{1}{2}s-c}}}{\sqrt{\text{ſin.}\,b\times\text{ſin.}\,c}}$, and,

$$\text{coſ.}\ \tfrac{1}{2}\ \text{angle} = \frac{r\sqrt{\text{ſin.}\,\frac{b+c-a}{2}\times\text{ſin.}\,\frac{b+c+a}{2}}}{\sqrt{\text{ſin.}\,b\times\text{ſin.}\,c}} = \frac{r\sqrt{\text{ſin.}\,\overline{\frac{1}{2}s-a}\times\text{ſin.}\,\frac{1}{2}s}}{\sqrt{\text{ſin.}\,b\times\text{ſin.}\,c}}$$

COROLLARY II.

167. Since we have, by *art.* 69, $\frac{\text{ſin.}\,A\times r}{\text{coſ.}\,A}$ = *tang.* A, it follows, that we ſhall alſo have, *tang.* $\frac{1}{2}$ an-

gle $=\frac{r\sqrt{\text{ſin.}\ \frac{1}{2}\overline{s-b}\times\text{ſin.}\ \frac{1}{2}\overline{s-c}}}{\sqrt{\text{ſin.}\ \frac{1}{2}\overline{s-a}\times\text{ſin.}\ \frac{1}{2}s}}$, and, $\text{cot.}\ \frac{1}{2}$ angle $=\frac{r\sqrt{\text{ſin.}\ \frac{1}{2}\overline{s-a}\times\text{ſin.}\ \frac{1}{2}s}}{\sqrt{\text{ſin.}\ \frac{1}{2}\overline{s-b}\times\text{ſin.}\ \frac{1}{2}\overline{s-c}}}$.

COROLLARY III.

168. Since we have, by *art.* 71, $\text{ſin.}\ 2\,A=\frac{\text{coſ.}\ A\times 2\,\text{ſin.}\ A}{R}$, we ſhall alſo have the *ſine* of the whole angle $A=\frac{2r\sqrt{\text{ſin.}\ \frac{1}{2}s\times\text{ſin.}\ \frac{1}{2}\overline{s-a}\times\text{ſin.}\ \frac{1}{2}\overline{s-b}\times\text{ſin.}\ \frac{1}{2}\overline{s-c}}}{\text{ſin.}\ b\times\text{ſin.}\ c}$; which formula may be otherwiſe found by a compariſon of the ſimilar triangles, *lOf*, LCF, &c.——The formula, $\text{coſ.}\ 2A=\frac{2\,\text{coſ.}^2\,A-R^2}{R}$, will give the *coſine*, likewiſe, of the whole angle $A=\frac{\overline{2\,\text{ſin.}\ \frac{1}{2}s\times\text{ſin.}\ \frac{1}{2}\overline{s-a}-\text{ſin.}\ b\times\text{ſin.}\ c}\times r}{\text{ſin.}\ b\times\text{ſin.}\ c}$; and, if we divide the former of theſe two expreſſions by the latter, we ſhall obtain an expreſſion, (though ſomewhat more complicated), for the *tangent* of the whole angle.

COROLLARY IV.

169. If the ſides, including the angle ſought, be equal, we ſhall have; $\text{ſin.}\ \frac{1}{2}$ angle $=\frac{R\times\text{ſin.}\ \frac{1}{2}BC}{\text{ſin.}\ AB}$, and, $\text{coſ.}\ \frac{1}{2}$ angle $=\frac{r\sqrt{\text{ſin.}\ \overline{b+\frac{1}{2}a}\times\text{ſin.}\ \overline{b-\frac{1}{2}a}}}{\text{ſin.}\ b}$.

SCHOLIUM.

170. From the ſeveral formulæ, which we have deduced from the two laſt Theorems, it will readily appear, that the reaſon of our chuſing to find

one of the angles of a triangle, whereof we know the three ſides, by the *ſine* or *coſine* of its half, is, becauſe theſe formulæ are the moſt eaſily conſtructed by the *logarithms*; however, it muſt be acknowledged, that the formulæ, which we have given for the *tangent* and *cotangent* of half the angle ſought, may be likewiſe very advantageouſly uſed, as being very little inferior in ſimplicity to the other two; and therefore, we have *four* eaſy and compendious methods of ſolving this particular caſe of Trigonometry.

Neper, *Oughtred*, *Jones*, and ſome other *Engliſh* Mathematicians, in order to preſerve a greater analogy between the ſolutions of this caſe by the *ſine* and *coſine* of half an angle, have given formulæ ſomewhat different from ours with regard to the expreſſions; which, as they ſeem pretty well adapted to remembrance, we ſhall here ſubjoin. Let *a* and *c* be put for the ſides, containing the angle ſought, the ſum whereof call, *s*, and difference, *d*, and let *b* repreſent the ſide oppoſite to this angle, (which we ſhall regard as the baſe), and we ſhall have;

$$\mathit{Sin.}\ \tfrac{1}{2}\ \text{angle} = \frac{r\sqrt{\mathit{ſin.}\frac{b+d}{2}\times\mathit{ſin.}\frac{b-d}{2}}}{\sqrt{\mathit{ſin.}\ a\times\mathit{ſin.}\ c}};\quad \mathit{coſ.}\ \tfrac{1}{2}\ \text{angle} = \frac{r\sqrt{\mathit{ſin.}\frac{s+b}{2}\times\mathit{ſin.}\frac{s-b}{2}}}{\sqrt{\mathit{ſin.}\ a\times\mathit{ſin.}\ c}};$$

and conſequently, *tang.* ½ angle

$$= \frac{r\sqrt{\mathit{ſin.}\frac{b+d}{2}\times\mathit{ſin.}\frac{b-d}{2}}}{\sqrt{\mathit{ſin.}\frac{s+b}{2}\times\mathit{ſin.}\frac{s-b}{2}}}.$$

If we would apply the preceding formulæ to plane trigonometry, it need only be conſidered that the *ſines* of the ſides of a triangle then become the ſides themſelves; and we ſhall have, by ſtill preſerving the ſame denominations for the three ſides of a triangle; *ſin.* ½ angle $= \frac{r\sqrt{\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}}{\sqrt{bc}}$; *coſ.* ½ angle $= \frac{r\sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}}}{\sqrt{bc}}$, and, *tang.* ½ angle $= \frac{r\sqrt{\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}}{\sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}}}$: from whence we may perceive, that if, to the logarithms of the factors of the numerator of each fraction, we add the arithmetical complements of the factors of the denominator, and take half the ſum thence ariſing, we ſhall obtain the angle ſought, much more ſimply than by the common rules of plane Trigonometry.

It may be moreover obſerved, that the formula found for the *ſine* of A, in *art.* 168, furniſhes us with a pretty remarkable property of a plane triangle; viz. that the *ſine* of any of its angles is equal to twice the area of this triangle, divided by the product of the ſides which include the ſaid angle: for it is demonſtrated, in all the Elements of Geometry, that the area of a triangle is $= \sqrt{\tfrac{1}{2}s\times\overline{\tfrac{1}{2}s-a}\times\overline{\tfrac{1}{2}s-b}\times\overline{\tfrac{1}{2}s-c}}$; which becomes the numerator of the formula for *ſin.* A, when the triangle is right-lined.

THEOREM V.

171. *In any ſpherical triangle,* BAC, *whereof the three angles are known, we ſhall always have theſe two analogies for finding one of its ſides;* 1°, *as the product of the ſines of the angles above the ſide ſought is to the product of the coſines of the differences of theſe angles and half the ſum of the three angles, ſo is radius ſquared to*

the ſquare of the coſine of half the ſide ſought; that is, *Sin.* B×*ſin.* C : *coſ.* $\overline{\frac{A+B+C}{2}-B}$ × *coſ.* $\overline{\frac{A+B+C}{2}-C}$: : R^2 : *coſ.*^{2} $\frac{1}{2}$BC; and 2°, *as the product of the ſines of the angles adjacent to the ſide ſought is to the product of the coſine of half the difference between theſe two angles and the third and half the ſum of the three angles, ſo is radius ſquared to the ſquare of the ſine of half the ſide ſought*; that is to ſay, *Sin.* B × *ſin.* C : *coſ.* $\frac{B+C-A}{2}$ × *coſ.* $\frac{B+C+A}{2}$: : R^2 : *ſin.*^{2} $\frac{1}{2}$ BC.

DEMONSTRATION.

Fig. 11. In the triangle DEF, all the parts whereof are the ſupplements to thoſe in the triangle BAC, by *art.* 130, we ſhall have, by *Theorem* III. *ſin.* FD×*ſin.* FE : *ſin* $\overline{\frac{DF+FE+DE}{2}-DF}$×*ſin.* $\overline{\frac{DF+FE+DE}{2}-FE}$: : R^2 : *ſin.*^{2} $\frac{1}{2}$ DFE. But, the arcs, FD, FE, are the ſupplements to the angles, B, C, in the triangle BAC; and conſequently, their *ſines* the ſame as thoſe of the ſaid angles. Moreover, ſince the *ſine* of half the ſupplement of any arc or angle is equal to the *coſine* of half ſuch arc or angle; the ſecond term will, after making proper ſubſtitutions, become, *coſ.* $\overline{\frac{B+C+A}{2}-B}$×*coſ.* $\overline{\frac{B+C+A}{2}-C}$. In like manner, the angle DFE being the ſupplement to the ſide BC, its half will be the complement of half this ſide, and the *ſine* of ſuch half the *coſine* of half the ſaid ſide; and therefore, the preceding proportion will now be changed into this; *Sin.* B × *ſin.* C : *coſ.* $\overline{\frac{B+C+A}{2}-B}$× *coſ.* $\overline{\frac{B+C+A}{2}-C}$: : R^2 : *coſ.*^{2} $\frac{1}{2}$ B C. Q. E. 1°. D.

2°. In the ſame triangle, DEF, we ſhall have, by *Theorem* IV. $\text{ſin. FD}\times\text{ſin. EF} : \text{ſin.}\,\frac{\text{DF}+\text{EF}-\text{DE}}{2}\times\text{ſin.}\,\frac{\text{DF}+\text{EF}+\text{DE}}{2} :: \text{R}^2 : \text{coſ.}^2\,\tfrac{1}{2}\,\text{DFE}$; which proportion, by making ſubſtitutions ſimilar to thoſe which we made in the preceding caſe, will be changed into, $\text{Sin. B}\times\text{ſin. C} : \text{coſ.}\,\frac{\text{B}+\text{C}-\text{A}}{2}\times\text{coſ.}\,\frac{\text{B}+\text{C}+\text{A}}{2} :: \text{R}^2 : \text{ſin.}^2\,\tfrac{1}{2}\,\text{BC}$. Q. E. 2°. D.

COROLLARY I.

172. Hence, if we call the ſum of the three angles of any triangle, s; theſe three angles, α, β, γ, reſpectively, and ſuppoſe, β, γ, to be the angles adjacent to the ſide ſought, we ſhall have the following formulæ; $\text{ſin.}\,\tfrac{1}{2}\,\text{ſide ſought} = \frac{r\sqrt{\text{coſ.}\,\frac{1}{2}s\times\text{coſ.}\,\overline{\frac{1}{2}s-\alpha}}}{\sqrt{\text{ſin.}\,\beta\times\text{ſin.}\,\gamma}}$; $\text{coſ.}\,\tfrac{1}{2}\,\text{ſide} = \frac{r\sqrt{\text{coſ.}\,\overline{\frac{1}{2}s-\beta}\times\text{coſ.}\,\overline{\frac{1}{2}s-\gamma}}}{\sqrt{\text{ſin.}\,\beta\times\text{ſin.}\,\gamma}}$, and, $\text{tang.}\,\tfrac{1}{2}\,\text{ſide} = \frac{r\sqrt{\text{coſ.}\,\frac{1}{2}s\times\text{coſ.}\,\overline{\frac{1}{2}s-\alpha}}}{\sqrt{\text{coſ.}\,\overline{\frac{1}{2}s-\beta}\times\text{coſ.}\,\overline{\frac{1}{2}s-\gamma}}}$. —— It would be eaſy to find formulæ for the *ſine*, *coſine*, *tangent*, and *cotangent*, of the whole ſide ſought; but, as theſe expreſſions would be much more complicated than thoſe above given, it will be unneceſſary to ſpecify them.

COROLLARY II.

173. When the two angles above the ſide ſought are equal, we ſhall have by the firſt part of the Theorem; $\text{ſin.}^2\,\text{B} : \text{coſ.}^2\,\tfrac{1}{2}\,\text{A} :: \text{R}^2 : \text{coſ.}^2\,\tfrac{1}{2}\,\text{BC}$,

and, by extracting the ſquare roots, $\text{coſ.}\ \tfrac{1}{2}BC = \frac{R\times\text{coſ.}\ \frac{1}{2}A}{\text{ſin.}\ B}$.——The formula, which would ariſe from the ſecond part, would not be quite ſo ſimple.

SCHOLIUM.

174. Theſe Theorems contain the ſolutions of all the poſſible caſes of oblique-angled ſpherical triangles, as we may be eaſily convinced from an inſpection of the ſecond *Table*, where all theſe ſolutions are collected and united. However, that we may give to this part all the extent and generality whereof it is ſuſceptible, we ſhall yet annex another Section; in which we ſhall give ſeveral other very important and general Theorems, and in particular the famous analogies of *Neper*; wherein this Geometer's deſign ſeems to have been, to reduce the analogies of ſpherical Trigonometry to thoſe of plane. But, by rendering to theſe Theorems all the generality requiſite, it will be eaſy to obſerve, that the analogies of plane, are but particular caſes of thoſe of ſpherical, Trigonometry; and therefore, ſufficient (in our opinion) to deduce them ſeparately from thence by ſimple inferences only. It is ſurpriſing that moſt writers upon Trigonometry have made no mention of theſe analogies; whilſt others have given them without demonſtration, and others deformed them, (but Mr. *Wolf* in particular), by ſubſtituting in their room proportions conſiderably different from thoſe of *Neper*.——Beſides, theſe analogies may prove to be of very peculiar ſervice in this; that they will greatly contribute to an eaſy retention of all the poſſible caſes of oblique-angled triangles, whether ſpherical or right-lined.

SECT. IV.

Containing a Demonſtration of the analogies of Neper, *as well as of ſome others, both in plane and ſpherical Triangles not commonly given.*

THEOREM I.

175. *If from any angle,* A, *of an oblique-angled ſpherical triangle,* BAC, *we let fall a perpendicular,* AD, *upon the oppoſite ſide or baſe,* B C, *(produced if neceſſary), we ſhall always have this analogy; as the tangent of half the baſe is to the tangent of half the ſum of the other two ſides, ſo is the tangent of half the difference of theſe ſides to the tangent of half the difference, or half the ſum, of the ſegments of the baſe formed by the perpendicular, according as it falls within or without the triangle*; that is to ſay, *Tang.* $\frac{1}{2}$BC : *tang.* $\frac{AB+AC}{2}$ :: *tang.* $\frac{AB-AC}{2}$: *tang.* $\frac{BD\mp DC}{2}$. Fig. 15.

DEMONSTRATION.

We have proved in *art.* 158 that, *coſ.* BD : *coſ.*CD :: *coſ.*AB : *coſ.*AC, and therefore, we ſhall alſo have, *componendo et dividendo*, *coſ.* BD+*coſ.*CD : *coſ.* BD—*coſ.* CD : : *coſ.* AB+*coſ.* AC : *coſ.* AB—*coſ.* AC; but, by *art.* 100, *coſ.* BD+*coſ.* CD : *coſ.* BD—*coſ.* CD : : *cot.* $\frac{BD+CD}{2}$:*tang.* $\frac{BD-CD}{2}$; and for the ſame reaſon, *coſ.* AB+*coſ.* AC : *coſ.* AB—*coſ.* AC : : *cot.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$: hence, ſince the two firſt terms of theſe two proportions are proportional, the two laſt will be ſo likewiſe, and give, *cot.* $\frac{BD+CD}{2}$: *tang.* $\frac{BD-CD}{2}$: : *cot.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$; which, after ſubſtituting for the *cotangents*

gents their values in the *tangents*, and multiplying extremes and means, produces, $\frac{\textit{tang.}\overline{\frac{1}{2}AB-\frac{1}{2}AC}}{\textit{tang.}\overline{\frac{1}{2}BD+\frac{1}{2}CD}} = \frac{\textit{tang.}\overline{\frac{1}{2}BD-\frac{1}{2}CD}}{\textit{tang.}\overline{\frac{1}{2}AB+\frac{1}{2}AC}}$. Now, if the perpendicular falls within the triangle, $\frac{BD+CD}{2}$ will be $= \frac{BC}{2}$, but if without, $\frac{BD-CD}{2} = \frac{BC}{2}$; and therefore, the laſt equation reduced into proportion will, to correſpond to both caſes, ſtand thus; *Tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB+AC}{2}$: : *tang.* $\frac{AB-AC}{2}$: *tang.* $\frac{BD\mp CD}{2}$. Q. E. D.

SCHOLIUM.

176. If we ſuppoſe the triangle right-lined, it is eaſy to perceive that, as the *tangents* then become the ſides themſelves, we ſhall have this proportion; *As the baſe of the triangle is to the ſum of the other two ſides, ſo is their difference to the difference or ſum of the ſegments formed by the perpendicular let fall upon the baſe, according as it falls within or without the triangle.*——It is likewiſe manifeſt, that this formula may be applied both to ſpherical and plane trigonometry, in order to reduce a triangle whoſe three ſides are given into two other right-angled ones; in each of which we ſhall know, beſides the right angle, the hypothenuſe and one leg. Yet (whether the triangle be ſpherical or right-lined) the formulæ in *art.* 164 and 165 ought (in our opinion) to be always preferred, as conducing to the eaſieſt and ſhorteſt ſolution.

THEOREM II.

Fig. 15. 177. *If we ſtill ſuppoſe that from any angle,* A, *of a ſpherical triangle,* BAC, *there be let fall a perpendicular,*

dicular, AD, *upon the oppoſite ſide, we ſhall have the two following proportions*; 1°, *as the ſine of the ſum of the angles above the ſide on which the perpendicular falls is to that of their difference, ſo is the tangent or cotangent of half the ſaid ſide to the tangent of half the difference, or cotangent of half the ſum, of its ſegments formed by the perpendicular*; that is, *Sin.* $\overline{C+B}$: *ſin.* $\overline{C-B}$: : *tang.* or *cot.* $\frac{1}{2}$ BC : *tang.* or *cot.* $\frac{BD\mp CD}{2}$; and 2°, *as the ſine of the ſum of the other two ſides is to that of their difference, ſo is the cotangent of half their included angle to the tangent of half the difference, or half the ſum, of the ſegments of this angle*; that is to ſay, *Sin.* $\overline{AB+AC}$: *ſin.* $\overline{AB-AC}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{BAD\mp CAD}{2}$.

DEMONSTRATION.

In the triangle BAC we have, by *art.* 158, *tang.* C : *tang.* B : : *ſin.* BD : *ſin.* CD, and therefore, we ſhall alſo have, *componendo et dividendo*, *tang.* C + *tang.* B : *tang.* C — *tang.* B : : *ſin.* BD + *ſin.* CD : *ſin.* BD — *ſin.* CD; but the relation of the two firſt terms is, by *art.* 87, equal to that of, *ſin.* $\overline{C+B}$: *ſin.* $\overline{C-B}$, and the relation of the two ſecond, by *art.* 95, equal to that of, *tang.* $\frac{BD+CD}{2}$: *tang.* $\frac{BD-CD}{2}$; therefore, if theſe relations be ſubſtituted for the preceding, and *tang.* $\frac{1}{2}$ BC be put for *tang.* $\frac{BD+CD}{2}$, when the perpendicular falls within the triangle; but for *tang.* $\frac{BD-CD}{2}$, when it falls without, we ſhall have for the two caſes; *Sin.* $\overline{C+B}$:

$\overline{C+B}$: *ſin.* $\overline{C-B}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{BD-CD}{2}$: : *tang.* $\frac{BD+CD}{2}$: *tang.* $\frac{1}{2}$ BC : : *cot.* $\frac{1}{2}$ BC : *cot.* $\frac{BD+CD}{2}$. Q. E. 1°. D.

2°. In the ſame triangle we ſhall likewiſe have, by *art.* 158, *tang.* AB : *tang.* AC : : *coſ.* CAD : *coſ.* BAD, and therefore, *componendo et dividendo, tang.* AB+*tang.* AC : *tang.* AB—*tang.* AC : : *coſ.* CAD +*coſ.* BAD : *coſ.* CAD—*coſ.* BAD : but, by *art.* 87, the relation of the two firſt terms is equal to that of, *ſin.* $\overline{AB+AC}$: *ſin.* $\overline{AB-AC}$, and that of the two ſecond, by *art.* 100, equal to that of, *cot.* $\frac{BAD+CAD}{2}$: *tang.* $\frac{BAD-CAD}{2}$; therefore, if we ſubſtitute theſe relations for the preceding, and put $\frac{1}{2}$ BAC for $\frac{BAD+CAD}{2}$, when the perpendicular falls within the triangle, but $\frac{1}{2}$ BAC for $\frac{BAD-CAD}{2}$, when it falls without, we ſhall get for the two caſes; *Sin.* $\overline{AB+AC}$: *ſin.* $\overline{AB-AC}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{1}{2}$ $\overline{BAD\mp\frac{1}{2}CAD}$. Q. E. 2°. D.

SCHOLIUM.

178. The firſt part of this Theorem, it is evident, conduces to the ſolution of a ſpherical triangle whereof one ſide and the two adjacent angles are given; by immediately reducing it into two right-angled ones, in each of which will be known one ſide and the adjacent angle. We might likewiſe uſe it for the ſame purpoſe in a right-lined triangle; only its application becomes unneceſſary for finding the ſides of ſuch a triangle, ſeeing they may be more eaſily obtained by the well known analogy

between the *ſines* of angles and their oppoſite ſides ; and therefore, the only caſe, wherein this Theorem can be of ſervice in plane Trigonometry, is that which requires the ſegments of the baſe of a triangle to be found. For, if we ſuppoſe the baſe and two adjacent angles given ; then, in order to find the ſegments thereof, we ſhall have this analogy ; *As the ſine of half the ſum of the angles above the baſe is to that of their difference, ſo is half the ſaid baſe to half the difference or half the ſum of its ſegments, according as the perpendicular falls within or without the triangle.*

The ſecond part of the Theorem may be applied to a ſpherical triangle, when there are given two of its ſides and the included angle, in order to reduce it into two right-angled ones, in each of which we ſhall know the hypothenuſe and an adjacent angle. It may likewiſe be applied to the correſponding caſe of right-lined triangles, in order to find the ſegments of the vertical angle, by ſubſtituting, $\overline{AB+AC}$. and $\overline{AB-AC}$, for the *ſines* of theſe quantities ; and in this caſe, the common rule will not be found much ſimpler.

THEOREM III.

179. *I ſay moreover, that in any ſpherical triangle,* BAC, *we ſhall have theſe two proportions* ; 1°, *as the ſine of half the ſum of any two angles is to that of half their difference, ſo is the tangent of half the ſide adjacent to theſe angles to the tangent of half the difference of their oppoſite ſides* ; or, which is the ſame, *Sin.* $\frac{C+B}{2}$: *ſin.* $\frac{C-B}{2}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB-AC}{2}$; and 2°, *as the coſine of half the ſum of the angles above one of the ſides is to that of half their difference, ſo is the tangent of half the adjacent ſide to that of half* *Fig.* 15.

the ſum of the oppoſite ſides; that is, *Coſ.* $\frac{C+B}{2}$: *coſ.* $\frac{C-B}{2}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{AB+AC}{2}$.

DEMONSTRATION.

We proved in the laſt Theorem that, *ſin.* $\overline{C+B}$: *ſin.* $\overline{C-B}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{BD-CD}{2}$; but, $C+B = \frac{2\times\overline{C+B}}{2}$, and, $C-B = \frac{2\times\overline{C-B}}{2}$; alſo, by the formula in *art.* 71, (*ſin.* 2A = 2 *ſin.* A × *coſ.* A), *ſin.* $\overline{C+B}$ will be found = 2 *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C+B}{2}$, and, for the ſame reaſon, *ſin.* $\overline{C-B}$ = 2 *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C-B}{2}$; and therefore, the foregoing proportion will, after dividing its two firſt terms by 2, be changed into the following; *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C+B}{2}$: *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C-B}{2}$: : *tang.* $\frac{1}{2}$ BC : *tang.* $\frac{BD-CD}{2}$. Moreover, ſince in every ſpherical triangle the *ſines* of the angles are to each other as the *ſines* of their oppoſite ſides, we ſhall have, *ſin.* C : *ſin.* B : : *ſin.* AB : *ſin.* AC, and conſequently, *componendo et dividendo*, *ſin.* C + *ſin.* B : *ſin.* C — *ſin.* B : : *ſin.* AB + *ſin.* AC : *ſin.* AB — *ſin.* AC; which proportion will, by ſubſtituting for its ſeveral terms their equals, as given in *art.* 91, 92, and 95, become, *ſin.* $\frac{C+B}{2}$ × *coſ.* $\frac{C-B}{2}$: *ſin.* $\frac{C-B}{2}$ × *coſ.* $\frac{C+B}{2}$: : *tang.* $\frac{AB+AC}{2}$: *tang.* $\frac{AB-AC}{2}$. Now, if we multiply this proportion by that which we before found, and expunge the contradictory quantities, we ſhall get, *ſin.*^{2} $\frac{C+B}{2}$: *ſin.*^{2} $\frac{C-B}{2}$: :

$\textit{tang.}\ \frac{1}{2} BC \times \textit{tang.}\ \frac{AB+AC}{2} : \textit{tang.}\ \frac{BD-CD}{2} \times \textit{tang.}\ \frac{AB-AC}{2}$; likewiſe, if we multiply the antecedents of the *alternate* proportion of the firſt Theorem by $\textit{tang.}\ \frac{AB+AC}{2}$, and the conſequents by $\textit{tang.}\ \frac{BD-CD}{2}$, we ſhall again get, $\textit{tang.}\ \frac{1}{2} BC \times \textit{tang.}\ \frac{AB+AC}{2} : \textit{tang.}\ \frac{BD-CD}{2} \times \textit{tang.}\ \frac{AB-AC}{2} :: \textit{tang.}^2\ \frac{AB+AC}{2} : \textit{tang.}^2\ \frac{BD-CD}{2}$; and therefore, ſince theſe two proportions have one common relation, we ſhall, by reaſoning *ex æquo*, obtain this analogy; $\textit{ſin.}^2\ \frac{C+B}{2} : \textit{ſin.}^2\ \frac{C-B}{2} :: \textit{tang.}^2\ \frac{AB+AC}{2} : \textit{tang.}^2\ \frac{BD-CD}{2}$, or, by extracting the roots, $\textit{Sin.}\ \frac{C+B}{2} : \textit{ſin.}\ \frac{C-B}{2} :: \textit{tang.}\ \frac{AB+AC}{2} : \textit{tang.}\ \frac{BD-CD}{2}$: : by *Theo.* I. $\textit{tang.}\ \frac{1}{2} BC : \textit{tang.}\ \frac{AB-AC}{2}$; from whence the truth of the firſt part of the Theorem manifeſtly follows.

2°. We might prove by a ſimilar calculation that, $\textit{Coſ.}\ \frac{C+B}{2} : \textit{coſ.}\ \frac{C-B}{2} :: \textit{tang.}\ \frac{1}{2} BC : \textit{tang.}\ \frac{AB+AC}{2}$: [*] for the whole art of this calculation conſiſts in diſpoſing the two proportions which we multiplied together in ſuch a manner, as to obtain the ſquares of the *coſines* of the quantities, C+B and C—B; a thing which is attended with very little difficulty. ——And hence the truth of the famous Theorem of *Neper* relating to both theſe caſes is clearly ſhewn.

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[*] Since by the firſt part of the Theorem, $\textit{coſ.}\ \frac{C+B}{2} \times \textit{ſin.}\ \frac{C+B}{2} : \textit{coſ.}\ \frac{C-B}{2} \times \textit{ſin.}\ \frac{C-B}{2} :: \textit{tang.}\ \frac{1}{2} BC : \textit{tang.}\ \frac{BD-CD}{2}$, and alſo, $\textit{coſ.}\ \frac{C+B}{2} \times \textit{ſin.}\ \frac{C-B}{2} : \textit{coſ.}\ \frac{C-B}{2} \times \textit{ſin.}\ \frac{C+B}{2} :: \textit{tang.}\ \frac{AB-AC}{2} : \textit{tang.}\ \frac{AB+AC}{2}$; therefore, by multiplying theſe

SCHOLIUM.

180. It is eaſy to perceive, that this Theorem may be applied in order to find the ſides of any ſpherical triangle, whereof the baſe and the angles above it are known. It might alſo be uſed for the ſame purpoſe in the correſponding caſe of plane Trigonometry, after ſubſtituting the baſe, and the ſum or difference of the ſides, for the *tangents* of theſe quantities.———It may likewiſe be obſerved, that the preceding demonſtrations carry along with them two other Theorems not altogether unworthy our regard. The firſt is this; *In*

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theſe two proportions together, we ſhall have, $cos.^2 \frac{C+B}{2} : cos.^2 \frac{C-B}{2} :: tang.\ \frac{1}{2} BC \times tang. \frac{AB-AC}{2} : tang. \frac{AB+AC}{2} \times tang. \frac{BD-CD}{2}$: but, from *Theo.* I. we ſhall get, $tang.\ \frac{1}{2} BC \times tang. \frac{AB-AC}{2} : tang. \frac{AB+AC}{2} \times tang. \frac{BD-CD}{2} :: tang.^2 \frac{AB-AC}{2} : tang.^2 \frac{BD-CD}{2}$; and conſequently, *ex æquo*, $cos.^2 \frac{C+B}{2} : cos.^2 \frac{C-B}{2} :: tang.^2 \frac{AB-AC}{2} : tang.^2 \frac{BD-CD}{2}$, or, $Cos. \frac{C+B}{2} : cos. \frac{C-B}{2} :: tang. \frac{AB-AC}{2} : tang. \frac{BD-CD}{2} :: tang.\ \frac{1}{2} BC : tang. \frac{AB+AC}{2}$.

any ſpherical triangle, the ſine of half the ſum of the angles at the baſe is to the ſine of half their difference, as the tangent of half the ſum of their oppoſite ſides is to the tangent of half the difference of the ſegments of the baſe; and the ſecond this; *The coſine of half the ſum of the angles at the baſe is to the coſine of half their difference, as the tangent of half the difference of their oppoſite ſides is to the tangent of half the difference of the ſegments of the baſe.*——Moreover, from this Theorem, and the conſideration that every right-lined triangle has the ſum of its three angles equal to two right ones, theſe two general analogies will ariſe; 1°, *As the ſine of half the vertical angle is to the coſine of half the difference of the other two angles, ſo is the baſe or ſide oppoſite to this angle to the ſum of the other two ſides*; that is to ſay, $\textit{Sin.}\ \frac{1}{2} A : \textit{coſ.}\ \frac{C-B}{2} :: BC : AB+AC$; and 2°, *as the coſine of half the vertical angle is to the ſine of half the difference of the other two angles, ſo is the baſe to the difference of the oppoſite ſides*; or, $\textit{Coſ.}\ \frac{1}{2} A : \textit{ſin.}\ \frac{C-B}{2} :: BC : AB-AC$. —— Laſtly, the two analogies, which we have obtained from the demonſtration of this Theorem, will give us this double analogy for any right-lined triangle, when the ſegments of the baſe are taken into conſideration; *As the ſum or difference of the ſides is to the difference of the ſegments of the baſe, ſo is the coſine or ſine of half the vertical angle to the ſine or coſine of half the difference of the angles above the baſe*; that is, $AB \pm AC : BD - CD :: \textit{coſ.}\text{ or }\textit{ſin.}\ \frac{1}{2} A : \textit{ſin.}\text{ or }\textit{coſ.}\ \frac{C-B}{2}$.

THEOREM IV.

181. *In every ſpherical triangle*, BAC, *we ſhall moreover have theſe two analogies*; 1°, *as the ſine of* *Fig.* 11.

half the ſum of the ſides including any angle is to that of half their difference, ſo is the cotangent of half the included angle to the tangent of half the difference of the other two angles; or, *Sin.* $\frac{AB+AC}{2}$: *ſin.* $\frac{AB-AC}{2}$: : *cot.* ½ BAC : *tang.* $\frac{C-B}{2}$; and 2°, *as the coſine of half the ſum of any two ſides is to that of half their difference, ſo is the cotangent of half their included angle to the tangent of half the ſum of their oppoſite angle*; that is to ſay, *Coſ.* $\frac{AB+AC}{2}$: *coſ.* $\frac{AB-AC}{2}$: : *cot.* ½ BAC : *tang.* $\frac{C+B}{2}$.

DEMONSTRATION.

In the triangle DEF, all the parts whereof are ſupplemental to thoſe in the triangle BAC, we ſhall have, by the preceding Theorem, *ſin.* $\frac{E+D}{2}$: *ſin.* $\frac{E-D}{2}$: : *tang.* ½ DE : *tang.* $\frac{FD-FE}{2}$. Now, in order to find what this analogy will become in the triangle BAC, let *a* and *b* be two circular arcs, repreſenting, for inſtance, the angles E and D, and let $2r$ be put $=180°$; then will the ſupplements to thoſe angles be, $2r-a$ and $2r-b$, and half the ſum of thoſe ſupplements, $2r-\frac{a}{2}-\frac{b}{2}$; that is to ſay, the ſupplement to half the ſum of the ſaid angles, *a* and *b*; conſequently, *ſin.* $\frac{E+D}{2}=$ *ſin.* $\frac{AB+AC}{2}$; and alſo, *ſin.* $\frac{E-D}{2}=$ *ſin.* $\frac{AB-AC}{2}$: laſtly, ſince *tang.* ½ ſupplement of an arc = *cot.* of half that arc, we ſhall have, *tang.* ½ DE = *cot.* ½ BAC, and, *tang.* $\frac{FD-FE}{2}=$ *tang.* $\frac{C-B}{2}$. Q. E. 1°. D.

We might ſhew exactly in the ſame manner, that the proportion, *coſ.* $\frac{E+D}{2}$: *coſ.* $\frac{E-D}{2}$: : *tang.* $\frac{1}{2}$ DE : *tang.* $\frac{DF+FE}{2}$, would be changed into this; *Coſ.* $\frac{AB+AC}{2}$: *coſ.* $\frac{AB-AC}{2}$: : *cot.* $\frac{1}{2}$ BAC : *tang.* $\frac{C+B}{2}$. Q. E. 2°. D.

SCHOLIUM.

182. We may perceive, by inſpection only, that theſe two analogies may be applied in order to find the two remaining angles of a ſpherical triangle, wherein two of the ſides and their included angle are known. We may likewiſe perceive, that the firſt analogy, when applied to the correſponding caſe of right-lined triangles, will give the common rule; *As the ſum of the ſides is to their difference, ſo is the cotangent of half their included angle to the tangent of half the difference of the oppoſite angles.* With reſpect to the ſecond analogy, it is manifeſt that, it cannot be applied to right-lined triangles, ſince *cot.* $\frac{1}{2}$ BAC then becomes equal to *tang.* $\frac{C+B}{2}$, becauſe in ſuch triangles the ſum of the three angles is neceſſarily equal to two right ones; which is a thing that can never prevail in ſpherical triangles, as before obſerved.——We might proceed to deduce other Theorems*, ſimilar to thoſe which we gave in the laſt Scholium; only we think it better to leave to Learners the pleaſure of finding theſe out by their own ingenuity.

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* They, who would ſee a variety of other properties, whether in plane or ſpherical Trigonometry, beſides thoſe which we have thought proper to ſpecify, may have recourſe to Mr. *Emerſon's* moſt comprehenſive and ſublime Treatiſe thereon; where we preſume they will find their curioſity abundantly ſatisfied.

TABLE I. *For the Reſolution of all the poſſible caſes of right-angled ſpherical Triangles.*

N. B. The letters S and A denote the caſes wherein the parts are *ſeparated* from, or *adjacent* to, the *mean*, according to the general Theorem of *Neper*, in *art.* 147.

Given.	*Required.*	*Values of the terms required.*		*Theorems.*	*Caſes wherein the Terms required are leſs than 90°*
Hypothenuſe and one leg.	Angle oppoſite to the given leg.	Its *ſin.* = $\frac{\text{ſin. given leg}}{\text{ſin. hypoth.}}$.	S . . .	*Th.* 1	If the given leg be leſs than 90°.
	Angle adjacent to the given leg.	Its *coſ.* = $\frac{\text{tang. given leg}}{\text{tang. hypoth.}}$.	A . . .	*Th.* 2	If the things given be of the ſame affection.
	Other leg.	Its *coſ.* = $\frac{\text{coſ. hypoth.}}{\text{coſ. given leg}}$.	S . . .	*Th.* 3	Idem.
One leg and its oppoſite angle.	Hypothenuſe.	Its *ſin.* = $\frac{\text{ſin. given leg}}{\text{ſin. given ang.}}$.	S . .	*Th.* 1	Ambiguous.
	Other leg.	Its *ſin.* = $\frac{\text{tang. given leg}}{\text{tang. given ang.}}$.	A . .	*Th.* 5	Idem.
	Other angle.	Its *ſin.* = $\frac{\text{coſ. given ang.}}{\text{coſ. given leg.}}$.	S . .	*Th.* 4	Idem.

One leg and the adjacent angle.	Hypothenuſe.	Its *tang.* = $\frac{\textit{tang. given leg}}{\textit{coſ. given angle}}$.	A	*Tb.* 2.	If the things given be of the ſame affection.
	Other angle.	Its *coſ.*=*coſ.giv.leg* × *ſin. giv. ang.*	S . .	*Tb.* 4.	If the given leg be leſs than 90°.
	Other leg.	Its *tang.*=*ſin.giv.leg* × *tang.giv.ang.*	A .	*Tb.* 5.	If the given angle be leſs than 90°.
Hypothenuſe and one angle.	Adjacent leg.	Its *tang.*=*tang. hyp.* × *coſ.giv.ang.*	A .	*Tb.* 2.	If the things given be of the ſame affection.
	Leg opp. to the giv. ang.	Its *ſin.*=*ſin. hyp.* × *ſin. giv. ang.*	S . .	*Tb.* 1.	If the given angle be acute.
	Other angle.	Its *tang.*=$\frac{\textit{cot. given angle}}{\textit{coſ. hypothenuſe}}$.	A	*Tb.* 6.	If the things given be of the ſame affection.
The two legs.	Hypothenuſe.	Its *coſ.*= *rectangle coſ. giv. legs.*	S . .	*Tb.* 3.	If the given legs be of the ſame affection.
	Either of the angles.	Its *tang.* = $\frac{\textit{tang. oppoſite leg}}{\textit{ſin. adjacent leg}}$.	A . . .	*Tb.* 5.	If the oppoſite leg be leſs than 90°.
The two angles.	Hypothenuſe.	Its *coſ.* = *rect. cot. giv. angles.*	A . .	*Tb.* 6.	If the angles be of the ſame affection.
	Either of the legs.	Its *coſ.*=$\frac{\textit{coſ. oppoſite angle}}{\textit{ſin. adjacent angle}}$.	S . . .	*Tb.* 4.	If the oppoſite angle be acute.

TABLE II. *For the Reſolution of oblique-angled ſpherical Triangles.*

N. B. We denote the firſt and ſecond ſegment of the angle or ſide, which is divided by a perpendicular, by I. and II. *ſeg.*

Given.	*Required.*		*Values of the Terms required.*
Two ang. and a ſide oppoſite to one of them.	The ſide opp. to the other angle.	*By the common analogy.*	*The ſin. of ang. are as the ſin. of their opp. ſides.*
	Third ſide.	*Let fall a per. upon the ſide contain'd betwixt the given ang.*	*Tang.* I. ſeg. of this ſide = *coſ. adj. ang.* × *tang. giv. ſide.* *Sin.* II ſeg. = $\frac{\text{ſin. I. ſeg.} \times \text{tang. ang. adj. to given ſide}}{\text{tang. ang. opp. to the ſaid ſide}}$.
	Third angle.	*Let fall a per. as before.*	*Cot.* I. ſeg. of this ang. = *coſ. giv. ſide* × *tang. adj. angle.* *Sin.* II. ſeg. = $\frac{\text{ſin. I. ſeg.} \times \text{coſ. ang. opp. to given ſide}}{\text{coſ. angle adj. to the ſaid ſide}}$.
Two ſides and an angle oppoſite to one of them.	The angle opp. to the other ſide.	*By the com. analogy.*	*The ſin. of ſides are as the ſin. of their oppoſite angles.*
	Angle included betwixt the giv. ſides.	*Let fall a perpen. from the included angle.*	*Cot.* I. ſeg. ang. ſought. = *tang. giv. ang.* × *coſ. adj. ſide.* *Coſ.* II. ſeg. = $\frac{\text{coſ. I. ſeg.} \times \text{tang. giv. ſide adj. to given ang.}}{\text{tang. ſide opp. to the ſaid angle}}$.
	Third ſide.	*Let fall a perpen. as before.*	*Tang.* I. ſeg. of this ſide = *coſ. giv. ang.* × *tang. of its adj. ſide.* *Coſ.* II. ſeg. = $\frac{\text{coſ. I. ſeg.} \times \text{coſ. ſide opp. given angle}}{\text{coſ. ſide adj. to the ſaid angle}}$.

Two ſides and the included angle.	An ang. opp. to one of the given ſides.	Let fall a perpen. from the third angle. { Tang. I. ſeg. of div. ſide = coſ. giv. ang. × tang. ſide opp. ang. ſought. Tang. ang. ſought = $\frac{\text{tang. giv. ang.} \times \text{ſin. I. ſegment}}{\text{ſin. II. ſeg. of div. ſide}}$.
	Third ſide.	Let fall a perpen. upon one of the giv. ſides. { Tang. I. ſeg. of div. ſide = coſ. giv. ang. × tang. other giv. ſide. Coſ. ſide ſought = $\frac{\text{coſ. ſide not div.} \times \text{coſ. II. ſegment}}{\text{coſ. I. ſeg. of ſide divided}}$.
A ſide and the two adjacent angles.	A ſide opp. to one of the giv. angles.	Let fall a perpen. upon the third ſide. { Cot. I. ſeg. of div. ang. = coſ. giv. ſide × tang. ang. opp. ſide ſought. Tang. ſide ſought = $\frac{\text{tang. giv. ſide} \times \text{coſ. I. ſeg. div. ang.}}{\text{coſ. II. ſeg. of the ſaid ang.}}$.
	Third angle.	Let fall a perpen. from one of the giv. ang. { Cot. I. ſeg. div. ang. = coſ. giv. ſide × tang. other giv. angle. Coſ. angle ſought = $\frac{\text{coſ. ang. not div.} \times \text{ſin. II. ſeg.}}{\text{ſin. I. ſeg. of div. angle}}$.
The three ſides.	*Let*, a, b, c, *be put for the three ſides, and*, s, *for their ſum; and let*, b *and* c, *be ſuppoſed to be the ſides which contain the angle ſought.*	
	An ang. by the ſin. or coſ. of its half.	Sin. $\frac{1}{2}$ ang. = $\frac{\sqrt{\text{ſin.}\frac{1}{2}s-b \times \text{ſin.}\frac{1}{2}s-c}}{\sqrt{\text{ſin.}b \times \text{ſin.}c}}$; coſ. $\frac{1}{2}$ ang. = $\frac{r\sqrt{\text{ſin.}\frac{1}{2}s \times \text{ſin.}\frac{1}{2}s-a}}{\sqrt{\text{ſin.}b \times \text{ſin.}c}}$.
The three angles.	*Let*, α, β, γ, *repreſent the three angles, the ſum whereof put* = s; *and let*, β *and* γ, *be ſuppoſed adjacent to the ſide required.*	
	A ſide by the ſin. or coſ. of its half.	Sin. $\frac{1}{2}$ ſide = $\frac{r\sqrt{\text{coſ.}\frac{1}{2}s \times \text{coſ.}\frac{1}{2}s-\alpha}}{\sqrt{\text{ſin.}\beta \times \text{ſin.}\gamma}}$; coſ. $\frac{1}{2}$ ſide = $\frac{r\sqrt{\text{ſin.}\frac{1}{2}s-\beta \times \text{ſin.}\frac{1}{2}s-\gamma}}{\sqrt{\text{ſin.}\beta \times \text{ſin.}\gamma}}$.

TABLE III.

For the Reſolution of all the poſſible caſes of oblique-angled ſpherical Triangles by the Analogies of Neper, *as demonſtrated in art. 175, &c.*

Given.	*Required.*	*Values of the Terms required.*
Two ſides and the included angle.	The other two angles.	$\text{Tang. }\frac{1}{2}\text{ their diff.} = \frac{\text{cot. }\frac{1}{2}\text{ giv. ang.} \times \text{ſin. }\frac{1}{2}\text{ diff. of giv. ſides}}{\text{ſin. }\frac{1}{2}\text{ ſum of theſe ſides}}.$ $\text{Tang. }\frac{1}{2}\text{ their ſum} = \frac{\text{cot. }\frac{1}{2}\text{ giv. ang.} \times \text{coſ. }\frac{1}{2}\text{ diff. of giv. ſides}}{\text{coſ. }\frac{1}{2}\text{ ſum of the ſaid ſides}}.$
	Third ſide.	*By the com. analogy. The ſin. of angles are as the ſin. of their oppo. ſides.*
Two ſides and an oppoſite angle.	Angle oppoſite to the other known ſide.	*By the com. analogy. The ſin. of ſides are as the ſin. of their opp. ang.*
	Third angle.	$\text{Cot. of its half} = \frac{\text{tang. }\frac{1}{2}\text{ diff. of other two ang.} \times \text{ſin. }\frac{1}{2}\text{ ſum giv. ſides}}{\text{ſin. }\frac{1}{2}\text{ diff. of theſe ſides}} = \frac{\text{tang. }\frac{1}{2}\text{ ſum of other two ang.} \times \text{coſ. }\frac{1}{2}\text{ ſum giv. ſides}}{\text{coſ. }\frac{1}{2}\text{ diff. of the ſaid ſides}}.$
	Third ſide.	*By the common analogy.*

Two angles above one of the ſides.	The other two ſides.	*Tang.* $\frac{1}{2}$ their diff. $= \frac{\text{tang.}\ \frac{1}{2}\ \text{giv. ſide} \times \text{ſin.}\ \frac{1}{2}\ \text{diff. giv. ang.}}{\text{ſin.}\ \frac{1}{2}\ \text{ſum of theſe angles}}$.
		Tang. $\frac{1}{2}$ their ſum $= \frac{\text{tang.}\ \frac{1}{2}\ \text{giv. ſide} \times \text{coſ}\ \frac{1}{2}\ \text{diff. giv. angles}}{\text{coſ.}\ \frac{1}{2}\ \text{ſum of the ſaid angles}}$.
	The third ang.	*By the com. analogy. The ſin. of ſides are as the ſin. of their opp. ang.*
Two angles and one of their oppoſite ſides.	Side opp. to the other giv. angle.	*By the common analogy.*
	Third ſide.	*Tang.* of its half $= \frac{\text{tang.}\ \frac{1}{2}\ \text{diff. given ſides} \times \text{ſin.}\ \frac{1}{2}\ \text{ſum oppo. angles}}{\text{ſin.}\ \frac{1}{2}\ \text{diff. of theſe angles}} = \frac{\text{tang.}\ \frac{1}{2}\ \text{ſum given ſides} \times \text{coſ.}\ \frac{1}{2}\ \text{ſum oppo. angles}}{\text{coſ.}\ \frac{1}{2}\ \text{diff. of the ſaid angles}}$.
	Third angle.	*By the common analogy.*
The three ſides.	One of the angles.	*Let fall a perpen. upon the ſide adja. to the ang. ſought.* *Tang.* $\frac{1}{2}$ ſum or $\frac{1}{2}$ diff. of the ſeg. of the baſe. $= \frac{\text{tang.}\ \frac{1}{2}\ \text{ſum} \times \text{tang.}\ \frac{1}{2}\ \text{diff. of the ſides}}{\text{tang.}\ \frac{1}{2}\ \text{baſe}}$.
		Coſ. angle ſought $=$ *tang. adja. ſeg.* $\times$ *cot. adja. ſide.*
The three angles.	One of the ſides	*Will be obtained by finding its correſpondent angle in a triangle, which hath all its parts ſupplemental to thoſe of the triangle, whoſe three angles are given.*

CHAP. III.

Containing the Graphical or Geometrical Reſolution of Spherical Triangles.

Obſervation upon the nature of theſe Solutions.

183. GEometers in general take very little notice of the *Graphical* or *Geometrical* ſolutions, and more eſpecially ſo, ſince logarithms were invented; conſidering, that the *Numerical* ſolutions are by their means rendered ſuſceptible of the greateſt ſimplicity, and that a much greater preciſion can be thence obtained, than can poſſibly be expected from conſtructions; becauſe, the preciſion in this caſe depends conjointly upon the art of the perſon who conſtructs, and the goodneſs of the inſtruments wherewith he conſtructs. We ſhould therefore have paſſed over this part entirely in ſilence, had it not been for the following conſiderations.

1°. *Geometrical* ſolutions, though not altogether exact for the reaſons juſt aſſigned, yet nevertheleſs appear ſtrictly ſo to the mind, as being founded upon the moſt approved and well-known truths of Geometry. 2°. There are ſeveral caſes, wherein all the preciſion of calculation is not abſolutely requiſite, and the facility of theſe operations very conſpicuous; as, when we would apply certain obſervations (whoſe accuracy we did not greatly regard) to the ſetting of a plan or walk according to ſome point of the compaſs; or to Dialling, in order to pave the way to a more perfect operation afterwards. 3°. There are Theories in Aſtronomy which are founded upon theſe ſolutions; and alſo, caſes of Pilotage, wherein they may be found of no inconſiderable uſe. They may likewiſe ſerve to conduct us in calculations, wherein we might be afraid of committing ſome miſtake; and, if we apply the Algebraical analyſis to them, we may be enabled to diſcover a great number of new ſolutions, which will not only ſerve to illuſtrate this part, but alſo lead us to ſome very elegant ſolutions of caſes in practical Aſtronomy, which would prove very complicated by other methods; as we ſhall endeavor to prove in the ſequel by their applications to various Problems. The formulæ, which we ſhall by this means obtain, will, if compared with the *Synthetic* ſolutions, point out to us ſome remarkable differences betwixt Syntheſis and Analyſis; and at the ſame time ſerve to increaſe the diligence of Learners, by putting them into a method of finding ſolutions deduced from Geometrical conſiderations. In the laſt place, it may be obſerved that, formulæ in general are much more ſimply and elegantly inveſtigated, when the calculations are founded upon Geometrical conſtructions, than when upon ſubſtitutions purely Algebraical; as will eaſily appear to any one, who will take the trouble of comparing our ſolutions with thoſe, which Mr. *De la Caille* hath given for the ſame formulæ.

PROBLEM I.

184. *Given two of the ſides,* AB, AC, *of a ſpherical triangle* BAC, *with their included angle,* A, *to find either of the angles above the baſe, and the baſe itſelf or third ſide,* BC. *Fig.* 17.

SOLUTION.

Upon the plane of the circle ABRD let there be taken the arc AB equal to one of the given ſides of the triangle BAC; to the centre G let the radii AG and BG be drawn, and through it the diameter *r*GR perpendicular to the radius AG; alſo, let there be taken on each ſide of the point A the arcs, AL, A*l*, equal to the given ſide AC, and through the extremities of theſe arcs the chord L*l* drawn: then, at the centre G let the angle DGR be made equal to the given angle BAC; and, letting fall from the point D the perpendicular D*d* upon the diameter R*r*, let CH be taken a fourth proportional to the lines, *r*G, *d*D, *l*H; and the point C will, as we have already obſerved in *art.* 160, be the projection of the angle C. This being done, in order to find one of the angles, as B for inſtance; through the point C draw the chord *f*CXF, as alſo the diameter MG*m*, perpendicular to the radius GB, and make, $Xf : XC :: GM : Gn$; then, through the point *n*, thus determined, draw the line *n*N perpendicular to GM, and terminated by the circumference at N, and the arc MN will be the meaſure of the angle B.——The angle C might be found by an exactly ſimilar operation, by taking AC upon the plane of the circle ABR*r* inſtead of the arc AB. Q. E. 1°. I.

2°. The third ſide BC is manifeſtly equal to either of the arcs, BF or B*f*; ſince the three arcs, B*f*, BC and BF, contain all the ſame number of degrees, as being compriſed betwixt the ſame point B, and a leſſer circle perpendicular to the radius GB. Q. E. 2°. I.

SCHOLIUM I.

185. The demonſtration of this conſtruction is an evident conſequence of the nature of the projection which is here made uſe of, and already explained in *art.* 160.——But, inſtead of repeating

the operation for obtaining the value of the angle C; we may likewiſe find it by the following eaſy method; which, in order to avoid too great an extent of the figures, we ſhall only barely point out. Through the extremities L and F of the arcs AL and BF, reſpectively equal to the ſides AC and BC, draw two *tangents* to meet the radii AG and BG, produced as far as neceſſary; then from the points of their interſection as centres, with theſe *tangents* as radii, deſcribe two circular arcs, and from the point where they cut each other draw two radii to the ſaid centres, and the angle which is contained betwixt them will be equal to the angle C; the reaſon of which conſtruction will evidently appear from this conſideration; that the angle which is contained betwixt two planes is equal to that which is formed by two lines perpendicular to their common interſection; as are in this caſe the *tangents* of the arcs AC and BC.

SCHOLIUM II.

186. Having taken, as before, the arcs, AL, A*l*, each equal to the ſide AC, the projection of the angle C may be determined without making uſe of proportionals, in the following manner. From the point H as centre, with the radius HL, deſcribe a ſemicircle L*cl*; alſo, at the ſame point make with the radius HL an angle LH*c* equal to the given angle BAC, and from the point *c*, where the radius H*c* cuts the circumference, let fall the perpendicular *c*C upon the diameter *l*L, and the point C will be the projection ſought of the angle C of the triangle BAC; only obſerve that, if the angle at A be obtuſe, the point C will fall upon H*l*, but, if acute, upon HL. —— The projection of the angle B may be found with equal fa-

cility: for if, upon the line *l*H, produced if neceſſary, C*x* be taken=CX, and, from the point *c* to *x*, the line *cx* be drawn, the angle *cx*L or *cxl* will be equal to the angle ABC.

Now in order to conceive the reaſon of this conſtruction, we muſt have recourſe to *fig.* 16, and we ſhall therein perceive, that the angle CHL is equal to the angle BAC, becauſe the lines, CH, HL, are both perpendicular to the common interſection, AG, of the planes, GAB, GAC; and conſequently, the angle *c*HL equal to the angle CHL. We ſhall alſo perceive, that in the triangle CX*c* (right-angled at *c*) the angle CX*c*, contained between the two lines, *c*X, CX, perpendicular to the common interſection, GB, of the planes, BGC, BGA, is equal to the angle ABC: but, according to our method of conſtruction, the right-angled triangle *c*C*x* in *fig.* 17, is perfectly equal to the triangle C*c*X in *fig.* 16, and of courſe the angle at *x* in both equal likewiſe.

SCHOLIUM III.

187. If we conſider the 17th figure with a little attention, we ſhall find that any triangle, ABC, determines therein three other triangles, which may be regarded as its correſpondents; viz. *ba*C, B*a*C and *b*AC; each of which hath one angle either common with or equal to one of the angles, and its other angles ſupplemental to the other two, in the triangle BAC; as alſo, two of its ſides ſupplemental to two of the ſides of the ſaid triangle, and one ſide equal, viz. that which is oppoſite to the equal angle: and as theſe triangles contain all the variations which the given parts of a Problem can poſſibly admit of, it may not be improper by way of exerciſe, and in order to underſtand more per-

fectly the nature of conſtructions, to make application of every particular caſe to each of them reſpectively.

PROBLEM II.

188. *Given two of the angles*, A, B, *of a ſpherical triangle*, BAC, *with their adjacent ſide*, AB; *to find*, 1°, *the other two ſides, and* 2°, *the third angle.* *Fig.* 17.

SOLUTION.

Having taken upon the circle ARD*ar* the arc AB equal to the ſide given, and drawn to the centre G the radii, AG, BG, erect the lines, G*r*, GM, reſpectively perpendicular thereto; then take the arcs, MN, RD, reſpectively equal to the angles at B and A, and having drawn the *ſines*, N*n*, D*d*, with the ſemi-axes, GB and G*n*, GA and G*d*, deſcribe the ſemi-ellipſes, B*nb*, A*da*; and their point of interſection, C, will be the projection of the angle C of the ſperical triangle BAC: then through this point let the chords, *l*CL, *f*CF, be drawn perpendicular to the radii, GA, GB, and they will give the arcs, AL, A*l*, equal to the ſide AC, and the arcs, BF, B*f*, equal to the ſide BC. Q. E. 1°. I.

2°. The ſides AC and BC being thus obtained, the angle C may be determined by either of the methods of ſolution given in the preceding Problem. Q. E. 2°. I.

PROBLEM III.

189. *Given two of the ſides*, AB, AC, *of a ſpherical triangle*, BAC; *with an angle oppoſite to one of them*; *to find* 1°, *the third ſide, and* 2°, *the other two angles.* *Fig.* 17.

SOLUTION I.

Let us ſuppoſe the angle given to be, B, and alſo that it is known of what affection the angle A is, which is included betwixt the given ſides, AB, AC. Then in the firſt place, take upon the plane of the circle ABR*ar* an arc, AB, equal to the given ſide AB, and draw the radii, GA, GB: which being done, through G draw MG*m* perpendicular to the radius GB, and take G*n* thereon equal to the *coſine* of the angle B towards M, if the angle BAC be obtuſe, or G*v* equal to the ſaid *coſine* towards *m*, if acute; and with, B*b*, *nv*, as axes, deſcribe the ellipſis B*nbv*. In the next place, take upon the circumference of the ſaid circle on each ſide of the point A the arcs, AL, A*l*, equal to the other given ſide AC; then draw the chord L*l*, and through the two points, C, C′, where it cuts the ellipſis B*nbv*, draw the lines, *f*CX, φC′*x*′, perpendicular to the radius GB; and they will give the arcs, B*f*, Bφ, equal to the third ſide BC, according as the oppoſite angle A is ſuppoſed to be obtuſe or acute. Q. E. 1°. I.

2°. Now in order to find the angle A; having deſcribed the ſemi-circle *l*KL upon the chord L*l* as diameter, and drawn the ordinates, C*c*, C′*c*′, draw alſo from the point H the lines, H*c*, H*c*′, and they will give the angles, LH*c*, LH*c*′, equal to angle A, according as it is obtuſe, or acute—The angle C may be found by the conſtruction in *art.* 185. Q. E. 2°. I,

SOLUTION II.

If we ſuppoſe the ſides AC and BC with the angle B oppoſite to the ſide AC to be given, we may ſolve the Problem by a different (though not more complicated than the preceding) method, as follows. Firſt, aſſume any point, B, upon the circumference of the circle ABR*ar* for the vertex of the angle B, and through this point and the centre G draw the diameter BG*b*; perpendicular to which draw the diameter MG*m*, and take *n*G thereon equal to the *coſine* of the given angle B; then taking on each ſide of the point B the arcs, BF, B*f*, equal to the given ſide BC, and drawing the chord F*f*, make, $MG : nG :: fX : XC$; and it is evident, that the point C (determined by this proportion upon the plane of the circle AR*ar*) will be the projection of the angle C. In the next place, through the centre G and point C draw the radius GCP, and upon the line GC as diameter deſcribe a circle; then, from the centre G with the radius GH, equal to the *coſine* of the given ſide AC, deſcribe an arc of a circle, and through the points, H, *h*, where it cuts the circle deſcribed upon GC, draw the lines, GHA, G*h*α; and they will give, AB, or αB, for the ſide required, according as the unknown angle oppoſite to the given ſide BC is ſuppoſed to be obtuſe or acute—We may likewiſe find the point C by a conſtruction exactly ſimilar to that given in *art.* 186: for, if we deſcribe a ſemi-circle upon the chord *f*F, draw through the point X a radius which may make with *f*X an angle equal to the angle B, and from the extremity thereof let fall a perpendicular upon the ſaid chord *f*F; the point C will by this means be determined independent of proportionals.— The angles A and C may be determined as in the laſt Problem. Q. E. I.

PROBLEM IV.

100. *Given two of the angles*, A, B, *of a ſpherical triangle*, BAC, *with a ſide*, AC, *oppoſite to one of them; to find*, 1°, *the ſide oppoſite to the other angle*; 2°, *the third ſide, and* 3°, *the third angle.* Fig. 17.

SOLUTION.

Through any point, A, upon the circumference of the circle ABR*ar* let the diameter AG*a* be drawn, and perpendicular thereto the diameter *r*GR. This done, on each ſide of the ſaid point let the arcs, AL, A*l*, be taken equal to the given ſide AC, and the chord L*l* drawn; upon which as diameter let the ſemi-circle *l*KL be deſcribed, and from the point H the radii, H*c*, H*c*′, drawn ſo as to make with LH an angle equal to the given angle A, according as it is obtuſe or acute: then from the points, *c*, *c*′, let fall the perpendiculars, *c*C, *c*′C′, and they will determine the points, C, C′, for the projections of the angle, C, or C′. In the next place, from the points, *c*, *c*′, let the lines, *cx*, *c*′*x*′, be drawn ſo as to make with the ordinates, C*c*, C′*c*′, the angles, C*cx*, C′*c*′*x*′, equal to the complement of the given angle B; this done, let the line GCP be drawn, and upon CG a circle, GHC*b*, deſcribed; after which, from the point C as centre with a radius equal to C*x* (equal, by *art.* 186, to CX, whoſe poſition will determine the ſides required) deſcribe a circular arc interſecting the former circle in two points X; then the ſide AB will be obtained by drawing the radius GXB; and a chord, *f*CXF, drawn through the points, X, C, will give the arcs, B*f*, BF, equal to the third ſide BC. ——The third angle may ſtill be determined by the conſtruction in *art.* 185. Q. E. I.

SCHOLIUM.

191. If the angles A and B be both acute, the ſide AB (and thence BC likewiſe) will be determined by means of the point X, which is found in the quadrant AG *r*. It is equally manifeſt that, if

we ſuppoſe a circle to be deſcribed upon C′G, and the line C′x' to be inſcribed therein, we ſhall obtain two other ſolutions of this Problem; viz. one, when the angle A in the triangle BAC is acute, and B obtuſe; and the other, when the angles at A and B are both acute, and that at C obtuſe.—We have omitted drawing the ſeveral lines neceſſary to the demonſtration, through fear of rendering the figure too confuſed.

PROBLEM V.

192. *Given the three ſides of a ſpherical triangle,* BAC; *to find one of its angles.* *Fig.* 17.

SOLUTION I.

Having taken upon the plane of the circle AR*ar* the arc AB equal to one of the given ſides, and drawn through the extremities thereof, A, B, the radii, AG, BG; take on each ſide of the point A the arcs, AL, A*l*, equal to the ſide AC, and alſo the arcs, B*f*, BF, equal to the third ſide BC; then draw the chords, L*l*, F*f*, and their point of interſection, C, will be the projection of the angle C. This done, draw through the centre G the diameters, *r*GR, *m*GM, perpendicular to the radii, GA, GB; then find a fourth proportional, G*n*, to the lines, *f*X, CX and MG, and through the point *n* (which will by this means be determined upon MG) draw the perpendicular *n*N, terminated by the circumference at N, and the arc MN will be equal to the meaſure of the angle B. In like manner, find a fourth proportional, G*d*, to the lines, *l*H, CH and *r*G, and through the point *d* (thus determined upon *r*G) draw the perpendicular *d*D, and it will give the arc RD equal to the angle at

A.

A.—The angle C might be found by an exactly ſimilar conſtruction, by taking the ſide AC upon the circumference of the circle ARar inſtead of AB. Q. E. I.

SOLUTION II.

If we would ſolve the Problem without making uſe of proportionals, it may be eaſily effected thus. Having found, as before, the point C by the interſection of the right lines, *l*L, F*f*, upon L*l* as diameter deſcribe the ſemi-circle LK*l*, and through the point C draw C*c* perpendicular to the ſaid diameter; then from the point *c* to H draw the line *c*H, and it will give the angle *c*HL equal to the angle at A. In the next place, take C*x*=CX, and draw the line *cx*, and the angle *cx*C will be equal to that at B.—The angle C might be found by a ſimilar conſtruction. Q. E. I.

PROBLEM VI.

193. *Given the three angles of a ſpherical triangle; to find one of its ſides.*

SOLUTION.

Fig. 11. Make a triangle DEF, which may have its three ſides ſupplemental to the angles of the given triangle; then, by one of the conſtructions in the preceding Problem, find the ſeveral angles of this triangle, and they will be the ſupplements to the ſides required. Q. E. I.

General Scholium for the foregoing ſolutions.

194. It is manifeſt that the ſix laſt Problems contain the Geometrical ſolutions of all the caſes of oblique-angled ſpherical triangles; but the con-

ſtructions may be equally (only with much greater ſimplicity) applied to right-angled triangles. We have omitted giving the demonſtration of every particular ſolution, ſeeing they may be all deduced as ſo many Corollaries from the general conſtruction of the orthographic projections already explained in *art.* 160, and laid down in *fig.* 16; the perfect underſtanding of which being once ſuppoſed, all the ideas neceſſary to a complete demonſtration of theſe ſolutions will follow of courſe.——We ſhall now ſubjoin ſome ſelect Problems relative to this kind of projection, as being that whereof the greateſt and moſt frequent uſe is made in Aſtronomy.

PROBLEM VII.

195. *Given the ellipſis which is the orthographic projection of a great circle, to find upon the plane of projection the appearance of the poles of this circle; and contrarily, having the projected poles of a great circle, to find the ellipſis which is the projection of the ſaid circle.*

SOLUTION.

Let us ſuppoſe the right line A*a* to repreſent the plane of the great circle of the ſphere, upon which the orthographic projection of all the points of the ſpherical ſurface is made, and that P*p*, perpendicular to GL, is the axis of the circle to be projected, which will conſequently be repreſented by GL; then it is evident, that the circle, AB*ab*, which paſſes through the poles, P, *p*, of the circle to be projected, will be perpendicular to the plane of projection. This being premiſed, if we let fall the perpendiculars, P*ϖ*, G*g*, it will eaſily appear that, *ϖ* will be the projection of the pole P, and *g* that of the extremity G of the diameter GL of the circle to be projected: but, ſince the angle GCP is a right *Fig.* 19.

one, the angle GCA will be the complement of the angle aCP, which denotes the elevation of the pole above the plane of projection; and therefore, it follows that, the diſtance $C\varpi$ of the projection of the pole P of a great circle C is equal to the *coſine* of the elevation of the ſaid pole above the plane of projection, or, which is the ſame, to the *ſine* of the elevation of the circle to be projected above the plane of projection; and that half the leſs axis of the ellipſis, which is the projection of this circle, is equal to the *ſine* of the elevation of the pole above the plane of projection. Q. E. I.

PROBLEM VIII.

196. *To find the dimenſions of the ellipſis, which is the orthographic projection of a leſſer circle of the ſphere.*

SOLUTION.

Fig. 19. Let the line MON repreſent a diameter of the leſſer circle, whereof the orthographic projection is required; alſo, let this line be perpendicular to the axis Pp, and the leſſer circle, which it repreſents, will be parallel to the circle GL, whereof Pp is the axis; then from the points, M, O and N, let fall the perpendiculars, Mm, Oo and Nn, upon the line Aa; and, it is manifeſt that, the point o will be the centre of the ellipſis, which is to be the projection of the leſſer circle MN parallel to GL, and mn the leſs axis thereof. This being premiſed, on account of the ſimilar triangles, $CP\varpi$, COo, we ſhall have, $CP : CO :: C\varpi : Co$; that is to ſay, *As radius is to the ſine of the diſtance of the leſſer circle from the great circle to which it is parallel, ſo is the coſine of the elevation of the pole of the ſaid great circle above the plane of projection to the diſtance* Co *of the centre* o *of the ellipſe to be deſcribed from the centre* C. The ſimilar triangles, GCg, MOR, likewiſe give, $GC : Cg :: MO : OR$ or $om = on$; that is, *As the ſine total is to the ſine of the elevation of the pole of the circle* GL *above the plane of projection, ſo is the ſine of the diſtance of the leſſer circle from the ſaid pole to half the conjugate axis of the ellipſis which is to be deſcribed*, and whereof the tranſverſe axis is manifeſtly equal to MN.——The point m might alſo be found by taking Cm equal to the *ſine* of the difference of the arcs, MG, BG; and conſequently the ellipſis ſought, deſcribed. Q. E. I.

Application of the two laſt Problems to the projections uſed in the Theory of Eclipſes.

197. The Earth being ſituate at a prodigious diſtance from the Sun, all the rays which come from thence hither may be conſidered as parallel; and if we imagine a plane perpendicular to the ray which iſſues from the centre of the ſun to that of the earth, this plane (which will cut the ſphere into two equal parts) will repreſent the diſk of the earth as appearing from the ſun; alſo ſeparate the enlightened from the darkened hemiſphere, and laſtly have all the points of the hemiſphere expoſed to the ſun projected orthographically upon it. This being premiſed, if, P, p, repreſent the poles of the world, the line GL will denote the equator; the little circle MN will be a parallel whereof GM is the Geographical latitude; the circle ABab will be the circle which Aſtronomers call the univerſal meridian; the line BC will repreſent the ray which comes from the centre of the ſun to that of the earth, and conſequently the arc BG be the ſun's declination; the line ACa will be a line perpendicular to the plane of the circle Bcb, which has the axis of the ecliptic projected upon it at the ſolſtices, and makes with the ſaid axis an angle equal to the obliquity of the ecliptic at the equinoxes; ACa will

will repreſent the circle upon which all the points of the earth ſeen from the ſun are orthographically projected, and laſtly, *a*P be the elevation of the pole above the illuminated diſk, and manifeſtly equal to the ſun's declination. Now if, after theſe definitions, we would deſcribe the ellipſis which is the projection of the parallel MN, it is evident that, the Problem will be exactly the ſame with that in the laſt article; except that the analogies, which we there gave for finding the centre *o* and half the leſs axis of the ellipſis to be deſcribed, will here be expreſſed thus: 1°. *As radius is to the ſine of the geographical latitude of the parallel, ſo is the coſine of the ſun's declination to the diſtance* C*o*; and 2°. *As the ſine total is to the ſine of the ſun's declination, ſo is the coſine of the geographical latitude of the parallel to half the conjugate axis of the ellipſis which is the projection of the ſaid parallel**.

198. As theſe graphical ſolutions very frequently occur in Aſtronomy, but particularly in the determination of the tranſits of the planets over the ſun's diſk, we ſhall here give the method of delineating upon the illuminated diſk the parallel of a place, whereof the latitude is known, according to the foregoing principles.

Fig. 20. Let AG*a*Q be the circle upon which the projection is to be made. Through any point, A, taken upon the circumference thereof, draw the diameter A*a*, and on each ſide of the ſaid point take the arcs, AP, A*p*, equal to the ſun's declination for the time when the projection is required: then draw the chord P*p*, and the point ϖ, where it cuts the diameter A*a*, will evidently be the projection of the pole P. In the next place, draw to the point P the radius CP; perpendicular to which draw the diameter ÆCQ, and it will repreſent the equator: then, from the points, Q, Æ, take the arcs, Q*l*, ÆL, equal to the latitude of the place (towards A if it be of the ſame denomination with the ſun's declination), and draw the chord L*l*, cutting the diameter P*p* in the point K; through which draw H*b* parallel to P*p*, and it will give O for the centre of the ellipſis to be deſcribed. This done, take CN equal to the *ſine* of the difference of the arcs, AP, ÆL, and ON will be half the leſs axis of the required ellipſis, whereof M*m* equal to L*l* is the greater axis: therefore, if we take O*n* = ON, and upon the axes, M*m*, N*n*, deſcribe an ellipſis MN*mn*, it will be the projection of the parallel given for the day when the ſun's declination is equal to the arc AP, as alſo touch the circle AG*a*Q in two points, G, I; through which if we draw the line GI (perpendicular to the radius CA), and reduce the arc GNI into hours, minutes and ſeconds, we ſhall get the length of the day for the ſaid parallel at the time when the declination is equal to the arc AP.

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* For further ſatisfaction ſee Mr. *De la Lande*'s Aſtronomie, p. 700, &c.

199. The projection of the equator upon the plane AG*a*Q may likewiſe be eaſily found; for if, from the extremities, Æ, Q, of the diameter ÆQ, which repreſents the equator, we let fall the perpendiculars, ÆS, Q*s*, and upon the axes, B*b*, S*s*, deſcribe the ellipſis BS*b s*, it will ſhew the projection of the terreſtrial equator as ſeen from the ſun at the time when the declination is equal to the arc AP.

200. We may alſo find upon the ſame plane the projection of the ecliptic, as well as of the equinoctial points. But in order to this it muſt be obſerved,

ſerved, that, as the axis of the equator makes always with that of the ecliptic an angle equal to the obliquity of the ecliptic, the pole P will deſcribe round the axis of the ecliptic the circumference of a leſſer circle, which will have its radius equal to the *ſine* of this obliquity, and be the baſe of a cone whoſe perpendicular height is equal to the *coſine* of the ſaid obliquity, taken upon the axis of the ecliptic: therefore, if upon the line Cϖ, as diameter, we deſcribe a ſemi-circle CVϖ, and inſcribe therein a line CV equal to the *coſine* of the obliquity of the ecliptic, and then draw the line VC*r*, we ſhall by this means manifeſtly obtain the axis of this great circle. Moreover, ſince the ſun is always in the plane of the ecliptic, it is evident that this circle muſt be repreſented by a right line; conſequently, if ECT be drawn perpendicularly to its axis RC*r*, this line will be the projection of the ſaid circle, and the points ♈, ♎, where it cuts the ellipſis BS*bs*, thoſe of the equinoctial points. In the next place, if through the point ♈ we draw the ordinate *f*♈F perpendicularly to the diameter B*b*, it will determine the arc *a*F equal to the ſun's right aſcenſion; and, if we alſo draw through the ſame point the line ♈D perpendicularly to the diameter ECT, the arc ♈D will be the meaſure of the ſun's longitude. We might proceed to find with equal facility the appearance of the ſolſticial points, both upon the equator and ecliptic; only we think it unneceſſary to inſiſt any longer upon this head, ſince it is ſufficient for our purpoſe to have ſhewn how eaſily the orthographic projections may be applied to the principal Problems of ſpherical Aſtronomy. We ſhall obſerve however, that ſeveral Aſtronomers ſeem not to entertain a ſufficiently juſt idea of the nature of theſe conſtructions; for, whilſt ſome reject them entirely, as deſtitute of a requiſite exactneſs, others there are on the contrary, who place too much dependance upon them: the uſe, which we think ought to be made of them, is that, which is generally made of all Geometrical figures; viz. to conduct the mind in the reſearch of numerical formulæ by the application of Algebra, or the general rules of Trigonometry: for which reaſon, and that the Theory of this kind of projections may be the better illuſtrated, we ſhall annex a few Corollaries upon the foregoing Problems.

COROLLARY I.

201. It follows from the laſt Problem, that in order to determine upon any ellipſis, which is the projection of a great circle, an arc which may contain any number of degrees, and alſo commence at a certain point upon this ellipſis; we need only let fall from the given point a perpendicular upon its tranſverſe axis, and produce it to the circumference of the circle deſcribed upon this axis as diameter; then, from the point where the circumference of the ſaid circle is cut by this perpendicular, ſet off an arc equal to the number of degrees given, and afterwards from the extremity thereof let fall another ordinate upon the tranſverſe axis of the ellipſis, and there will by this means be determined upon the periphery of the ellipſis an arc containing the number of degrees aſſigned.

COROLLARY II.

202. If the arc is to contain 90°: having let fall from the given point a perpendicular upon the tranſverſe axis of the ellipſis, we need only take on the ſaid axis, on the other ſide of the centre, a line

equal to the *ſine* of the arc comprehended betwixt that extremity of the axis which is neareſt to the given point, and the perpendicular or ordinate which paſſeth through this point. For example, if we would determine upon the ellipſis AC*da* an arc of 90°, commencing at the point C; we need only take the line Gλ equal to the *ſine l*H of the arc A*l*=AC, terminated by the ordinate which paſſes through the point given, and then at the point λ, thus obtained, erect the perpendicular λ*l'*, and it will give upon the ſaid ellipſis the arc C*l'* equal to 90°. In like manner, if, upon half the greater axis G*b* of the ellipſis BC*b*, we take the line Gφ equal to the *ſine* FX of the arc BF=BC, and erect the perpendicular φ*f'*, we ſhall get upon this ellipſis the arc C*f'*= 90°. *Fig.* 21.

COROLLARY III.

203. From what has been already ſaid, it will be very eaſy to find upon the plane of the circle ABR*ar* the meaſure of the angle C or BCA; for, as every ſpherical angle is meaſured by the arc of a great circle compriſed betwixt its ſides, and deſcribed from its vertex as pole, it is manifeſt that the whole difficulty will conſiſt in deſcribing the ellipſis Q*l'q*, which is the projection of the great circle whoſe pole is at C; the method whereof is as follows. Through the point C and centre G draw the diameter PCG*p*, and perpendicular thereto through the ſame points the lines, G'C*g'*, QG*q*; then take upon G*p* a part, GS, equal to G'C, which is the *ſine* of the elevation of the point C above the plane of projection, and this line will be half the leſs axis of the ellipſis which is the projection of the great circle whoſe pole is at C: then the points, *l'*, *f'*, where this ellipſis cuts the ellipſes, A*da*,

B*nb*, will determine the arc fl', whoſe correſpondent, $\beta\gamma$, upon the circle ABR*ar*, contained betwixt the ordinates, $\gamma\lambda$, $\beta\phi$, will be the meaſure of the angle at C.

COROLLARY IV.

204. But, in order to find the meaſure of the angle C, it may be eaſily perceived that, it is not abſolutely neceſſary to deſcribe the ellipſis which is the projection of the great circle whereof this angle is the pole: for, having determined, as before, the points, l', f', upon the ellipſes, AC*a*, B*nb*, if from theſe points we let fall upon the diameter QG*q*, perpendicular to GCP, the ordinates, $l'\lambda$, $f'\phi$; their prolongations to the circumference of the circle AR*ar* will determine the arc $\beta\gamma$ equal to the meaſure of the angle C.

PROBLEM IX.

205. *To find upon the ſurface of the circle* ABR*ar*; *Fig.* 21. 1°, *the projection of the arc* Aδ *let fall from the angle* A *perpendicularly upon the oppoſite ſide* BC; 2°, *the values of the ſegments*, Bδ, Cδ, *of the ſide* BC; 3°, *the ſegments of the angle* BAC, *and* 4°, *the value of the perpendicular* Aδ.

SOLUTION.

Since the arc, whoſe projection we ſeek, is part of a great circle of the ſphere paſſing through the point A, it is manifeſt that the line AG*a* will be the tranſverſe axis of the ellipſis which is the projection thereof: moreover, ſince this circle is to be perpendicular to the ſide BC, it muſt neceſſarily paſs through the pole of this arc, and of conſequence the ellipſis required alſo through the projection of the ſaid pole: therefore, if we take upon the radius

Gm a part GZ=Nn, the point Z will, by *art.* 195, be the pole of the arc BC, as well as one of the points of the ellipſis to be deſcribed; then, if through this point Z we draw a line Zζ perpendicularly to the diameter Aa, and produce it till it meets the circumference of the circle ARa in a point θ, and likewiſe take upon RG a line, Gρ, a fourth proportional to the lines θζ, ζZ and RG; the ellipſis Aδρa deſcribed upon the ſemi-axes GA, Gρ, will be the required projection of the circle perpendicular to BC. Q. E. 1°. I.

2°. If through the point, δ, where this ellipſis cuts the ſide BC, δμ be drawn perpendicularly to the radius GB, the arcs, Cμ, Bμ, will be the ſegments of the ſide BC. Q. E. 2°. I.

3°. If at the point μ, we raiſe μo perpendicularly to the radius GR, the arcs Ro, Do, will be the meaſures of the ſegments, BAδ, CAδ, of the angle BAC. Q. E. 3°. I.

4°. Laſtly, if through the point δ we draw the ordinate δν perpendicularly to the radius GA, the arc Aν will be the value of the perpendicular Aδ. Q. E. 4°. I.

SCHOLIUM I.

206. If it was required to determine the projection of the circle, which ſhould divide the angle BAC into two equal parts, it might be effected by an exactly ſimilar conſtruction; by taking upon the radius GR a line, Gk, equal to the *coſine* of half the angle BAC, and deſcribing an ellipſis upon GA and Gk as ſemi-axes.——If the perpendicular was ſuppoſed to fall from the angle C upon the ſide AB, produced as far as neceſſary, the Problem might then be ſolved by means of the conſtruction given in *art.* 203. Laſtly, it is manifeſt, from the conſtruction of this Problem, that the arc Rθ is

equal to the arc Aδ, perpendicular to the ſide BC, ſince the arcs, AR, δθ, each 90°, have the part δR common.

SCHOLIUM II.

207. This conſtruction naturally leads us to a very ſimple and very elegant demonſtration of one of the famous Analogies of *Neper*, nearly reſembling that which he himſelf hath given us in his Treatiſe entitled, "*Logarithmorum mirifici Canonis Conſtructio.*" Having determined the ſeveral parts ſpecified in the foregoing Problem by the method above given, at the extremity, A, of the diameter Aa erect the indefinite perpendicular AΛ; then from the point B as pole, with a circular radius equal to the arc BC, deſcribe a leſſer circle upon the ſurface of the ſphere, and it will manifeſtly paſs through the points, L, C, c and l: moreover, through the points, C, c, draw the right lines, FC*f*, ω*co* parallel to the *tangent* AK; and it is evident that the arc A*f* will be equal to the arc AC, as alſo the arc A*o* equal to the arc A*c*, which is the difference of the ſegments, Aδ, δC, formed upon the ſaid arc AC by the perpendicular Bδ. This being premiſed, if we in the next place draw from the point *a* through the points, L, *l*, *f* and *o*, the right lines, *a*LΛ, *al*λ, *af*K and *aok*, and produce them to the *tangent* AΛ; it will eaſily appear that the four points, Λ, λ, K and *k*, will be in the circumference of a circle deſcribed upon the plane repreſented by AΛ, and touching the ſphere at A, and that this circle is the baſe of an oblique cone whoſe vertex is at *a*. To demonſtrate which, it will be ſufficient to prove that the ſection of a cone, which has for its baſe the circle deſcribed upon the diameter L*l*, by the plane AΛ, is antiparallel (ſubcontrary) to the ſaid baſe; ſince it is ſhewn

Fig. 22.

in ſeveral Treatiſes upon Conics that ſuch a ſection is circular. Now the angle AΛ*a*, being formed by a *tangent* and *chord*, will be meaſured by half the difference of the arcs, AF*a*, AſL, compriſed between its ſides: but, AL=AI, and therefore, AF*a*—AſL=AF*a*—AFI=*a*I=*a*L; whence, as the half of this laſt arc is exactly the meaſure of the angle *al*L, the triangles, *al*L, *a*Λλ, will be ſimilar: conſequently, the cone will be cut ſub-contrarily to its baſe, and of courſe the circle deſcribed upon *l*L projected into a circle upon the plane AΛ. This proved, the interior *ſecants* will laſtly give, AK: AΛ::Aλ: A*k*; that is, *As the tangent of half the baſe,* AC, *is to the tangent of half the ſum of the ſides,* AB *and* BC, *ſo is the tangent of half the difference of theſe ſides to the tangent of half the difference of the ſegments of the baſe.*

In order to be convinced of the identity of theſe two proportions, it muſt be obſerved that, by regarding the diameter A*a* as radius, the lines, AK, AΛ, Aλ and A*k*, will be the *tangents* of the angles, A*a*K, A*a*Λ, A*a*λ and A*ak*, at the point *a*; and that, as theſe angles have their vertices in the circumference of the circle AL*a*, they will be meaſured by the halves of the arcs contained betwixt their ſides: but, the arc Aſ is, by conſtruction, equal to the baſe AC of the ſpherical triangle BAC; the arc AL to the ſum of the ſides AB and BC; A*l* to the difference of theſe ſides, and A*o* to the difference of the ſegments of the baſe, ſince the arcs, A*e*, A*o*, are contained betwixt the parallel planes, AΛ, *eo*, and conſequently equal.

Scholium III.

208. From the preceding demonſtration it may be collected that two kinds of projections have

been made uſe of; the one *orthographic* of all the parts of the triangle upon the plane of the circle AL*a*; and the other upon the plane repreſented by A*λ*, which is called the *polar* projection, becauſe the eye was ſuppoſed to be placed at the point *a*, conſidered as one of the poles of the ſphere. Now *theſe* projections have a great affinity with the *ſtereographic* ones (whereof very frequent uſe is made with reſpect to *Maps*), and differs from them only in this; that in the *latter*, the eye may be ſuppoſed at *any* point upon the ſurface of the ſphere, and the projection of the oppoſite hemiſphere made upon the plane of a great circle, produced as far as neceſſary, and perpendicular to the radius drawn from the centre of the ſphere to the place of the eye upon the ſphere's ſurface. The pre-eminence of theſe projections ariſes from hence, that they always give ſimilar figures to thoſe which are required to be projected, and conſequently repreſent upon a plane the Map of a Country by figures ſimilar to thoſe which are formed upon the ſurface of the globe: which important property is a neceſſary conſequence of this, that the cones of projections are always cut in a ſub-contrary manner by the plane of projection. They who would thoroughly underſtand this ſubject, may have recourſe to the works of Father *Tacquet*, or to the Optics of Father *Aiguillon*.——We ſhall now, in the laſt place, ſubjoin the following Problem, ſince it furniſhes us with an extremely eaſy ſolution of moſt of the caſes of ſpherical Trigonometry, as alſo with a new ſort of graphical ſolution.

PROBLEM X.

209. *Given the three ſides of a ſpherical triangle,* BAC; *to find one of the angles thereof, by developing its parts.*

SOLU-

SOLUTION.

Fig. 23.

Upon any circle, BAC, take the three arcs, AB, AC and BC, reſpectively equal to the three ſides of the triangle whoſe angles are required; and having drawn the radius GA (if it be the angle A which we would find), draw thereto the *tangent* *d*AD, terminated at *d* and D by the prolongations of the radii GC and GB; then upon the radius GC, produced as far as neceſſary, take the line G*d* equal to the *ſecant* G*d* of the ſide AC; and from the points, D, A, as centres, with the radii, D*d*, A*d*, deſcribe two circular arcs interſecting each other in the point δ, and they will give the angle δAD equal to the angle A of the triangle BAC.— We might find by a ſimilar conſtruction the other two angles of this triangle.

This ſolution, it is evident, carries with it its own demonſtration; for, the lines, AD, A*d* or Aδ, being the *tangents* of the arcs, AB, AC, and by conſtruction perpendicular to the common interſection, AG, of the planes, BAG, CAG, the angle formed between them, when terminated by the line Dδ=D*d*'†, will manifeſtly be equal to the angle A in the triangle BAC†. Q. E. I. and D.

† Equal to the line intercepted between the ſecants of the arches AC & AB paſsing through the extremities of the arch BC.

COROLLARY I.

210. This conſtruction, it is very plain, might be made uſe of in order to find the ſides of a triangle whereof the three angles are given, by applying it to a triangle whoſe ſeveral parts are ſupplemental to thoſe of the ſaid triangle (*art.* 130).

COROL-

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† Should not this explanation appear ſufficiently clear and ſatisfactory, conſult the firſt Theorem of *Simpſon*'s Trigonometry, p. 23.

COROLLARY II.

21. It is equally apparent that this conſtruction may be likewiſe applied to the ſolution of a triangle, whereof two ſides and the included angle are known. For, having taken the arcs, AB, AC, reſpectively equal to the ſides containing the given angle, and drawn the *tangents*, AD, Ad, terminated by the radii, GB, GC, produced as far as neceſſary; we need only make the angle DAδ equal to the angle given, take $A\delta$ equal to the *tangent* Ad, and draw $D\delta$; then, from the points G and D as centres, with the radii $Gd' = Gd$ and $Dd' = D\delta$, deſcribe two circular arcs, and afterwards from the point of their interſection, d', draw to the centre G the line $d'G$, and the arc BC will by this means be determined equal to the third ſide; from whence the other angles may be eaſily obtained.

CHAP. IV.

Containing the Analytical or Algebraical Reſolution of ſpherical Triangles.

ADVERTISEMENT.

ALL the calculations, which we ſhall have to make in this Chapter, being founded upon the conſtructions given in the preceding one, it will be abſolutely neceſſary to underſtand *them* perfectly before we proceed any further. However, that theſe calculations may not be rendered too difficult for Learners by frequent changes of the ſigns, we ſhall uſually apply them to the 17th *fig.* the ſeveral parts whereof we ſhall denote as in the following Table: but, if any terms happen to change their ſigns when the formulæ by this means derived are applied to other triangles, we are from thence to conclude that the angles, which were before ſuppoſed acute or obtuſe, become then obtuſe or acute; or that the ſame changes are made with reſpect to the ſides.

TABLE.

Let the *radius* be put = *r*, and the parts of the triangle BAC expreſſed thus;

Sin. AB = BK = a. *Coſ.* BAC = Gd = q.

Coſ. AB = GK = b. *Sin.* ACB $h = \frac{ap}{c} = \frac{am}{f}$.

Sin BC = FX or fX = c. *Coſ.* ACB = $k = \sqrt{\frac{r^2c^2 - a^2p^2}{c^2}}$

Coſ. BC = GX = d. *Tang.* AB = $\frac{ar}{b}$.

Sin. AC = LH or lH = f. *Tang.* BC = $\frac{cr}{d}$.

Coſ. AC = GH = g. *Tang.* AC = $\frac{fr}{g}$.

Sin. ABC = Nn = m. *Tang.* B = $\frac{mr}{n}$.

Coſ. ABC = Gn = n. *Tang.* A = $\frac{pr}{q}$.

Sin. BAC = Dd = p. *Tang.* C = $\frac{hr}{k}$.

Then the conſtructions in *art.* 160 will give the following values of the lines, CX, Cf, &c.

CX = $\frac{cn}{r}$. CH = $\frac{fq}{r}$.

Cf = $c - \frac{cn}{r}$. CL = $f + \frac{fq}{r}$.

CF = $c + \frac{cn}{r}$. Cl = $f - \frac{fq}{r}$.

For greater expedition and certainty, and to prevent the trouble of recurring inceſſantly to this Table in order to find the analytical expreſſions of the quantities it contains, it will not be improper for the Reader to tranſcribe the whole upon a piece of looſe paper, and carry the ſame along with him as he proceeds in the ſolution of the ſeveral Problems.

PROBLEM I.

212. *To find the relation betwixt the ſines of the angles*

angles and thoſe of the ſides of any ſpherical triangle.

SOLUTION.

From the interſection of the chords, *f*F, *l*L, we ſhall have, CF × C*f* = CL × C*l*, or, which amounts to the ſame,* $\overline{fX}^2 - \overline{CX}^2 = \overline{lH}^2 - \overline{CH}^2$; and, if for these quantities their algebraical values be ſubſtituted, we ſhall get, $c^2 - \frac{c^2n^2}{r^2} = f^2 - \frac{f^2q^2}{r^2}$, or, $c^2r^2 - c^2n^2 = f^2r^2 - f^2q^2$; which, after putting m^2 in the place of $r^2 - n^2$, and p^2 for $r^2 - q^2$, become, $c^2m^2 = f^2p^2$; then, if we extract the ſquare roots of the members of this equation, and put theſe roots into proportion, we ſhall find, $c : f :: p : m$; that is, *Sin.* BC : *ſin.* AC : : *ſin.* BAC : *ſin.* ABC : from whence it is manifeſt that in any ſpherical triangle, *The ſines of the angles are to each other as thoſe of their oppoſite ſides.* Q. E. I.

Fig. 17.

PROBLEM II.

213. *Given two ſides and their included angle; to find* 1°, *either of the other angles, and* 2°, *the third ſide.*

SOLUTION.

Let the ſides, AB, AC, with the included angle BAC be the parts given, and let it be firſt required to find the angle B. ——— On account of the ſimilar triangles, GKB, GHO, we ſhall have, GK : KB : : GH : HO, or, $b : a :: g : \frac{ag}{b}$; alſo, by the conſtruction given in *art.* 160 we ſhall have, G*r* : G*d* : : *l*H : CH, or, $r : s :: f : \frac{fs}{r}$; therefore, CO or CH

Fig. 17.

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* This will eaſily appear from the 5th Propoſition of the 2d Book of *Euclid*.

$CH+HO=\frac{fq}{r}+\frac{ag}{b}$. Moreover, it is evident that the angle OCX in the right-angled triangle CXO is equal to the angle AGB; wherefore, we ſhall get from ſimilarity of triangles, R : *coſ.* AB :: CO : CX; and conſequently, $CX=\frac{bfq}{rr}+\frac{ag}{r}$; laſtly, ſince the lines, *f*X, CX, MG and *n*G, are proportional by conſtruction, we ſhall have *f*X : CX :: MG : *n*G, or analytically, $\frac{fp}{m}$ (by *art.* 212) : $\frac{bfq}{rr}+\frac{ag}{r}$ :: *r* : *n*; from whence we get, $\frac{nfp}{m}=\frac{bfq}{r}+ag$, and therefore, $\frac{m}{n}=\frac{fp}{bfq+agr}$ (by dividing the ſecond member of the equation by *f*) $\frac{p}{bq+\frac{agr}{f}}$; that is to ſay, $Tang.\,B=\frac{RR\times fin.\,A}{fin.\,AB\times cot.\,AC+coſ.\,AB\times coſ.\,A}$ Q. E. 1°. I.

2°. In order to obtain the third ſide BC, we need only regard the *ſine* of this ſide as unknown, and put *c* in the place of $\frac{fp}{m}$ in the equation $\frac{nfp}{m}-ag=\frac{bfq}{r}$, and we ſhall get, $c=\frac{ag}{n}+\frac{bfq}{rn}$; that is, *Sin.* $BC=\frac{coſ.\,AC\times fin.\,AB}{tang.\,B}+\frac{coſ.\,AB\times coſ.\,A\times fin.\,AC}{R\times tang.\,B}$. Q. E. 2°. I.

COROLLARY.

214. It is evident, that if we ſuppoſe different ſides with the included angle given, and purſue a ſimilar proceſs to that whereby we found the angle B, we ſhall get the following formulæ;

These Theorems suppose ~~[illegible]~~ the opp. b. giv[illegible]
to be greater than 90°

$$Tang.\ B = \frac{R^2 \times sin.\ A}{sin.\ AB \times cot.\ AC + cos.\ AB \times cos.\ A} = \frac{R^2 \times sin.\ C}{sin.\ BC \times cot.\ AC + cos.\ BC \times cos.\ C}$$

$$Tang.\ C = \frac{R^2 \times sin.\ A}{sin.\ AC \times cot.\ AB + cos.\ AC \times cos.\ A} = \frac{R^2 \times sin.\ B}{sin.\ BC \times cot.\ AB + cos.\ BC \times cos.\ B}$$

$$Tang.\ A = \frac{R^2 \times sin.\ C}{sin.\ AC \times cot.\ BC + cos.\ AC \times cos.\ C} = \frac{R^2 \times sin.\ B}{sin.\ AB \times cot.\ BC + cos.\ AB \times cos.\ B}$$

But, here it must be obſerved that, ſome of the terms of theſe formulæ will have their ſigns changed, according to the different combinations of acute or obtuſe angles; ſo that whenever we would apply them to any particular caſe, we muſt have reſpect to this conſideration. For inſtance, if we would obtain the value of the angle B of the triangle BAC in *fig.* 18, from having the ſides, AB, AC, and their included angle A given, we ſhall find, as in *fig.* 17, that HO $= \frac{ag}{r}$, and CH $= \frac{fq}{r}$; but HQ is equal to HO − CH, and therefore the *Tang.* of the angle B becomes $= \frac{R^2 \times sin.\ A}{sin\ AB \times cot.\ AC - cos.\ AB \times cos.\ A}$. In like manner the term *coſ.* AC × *coſ.* A would become negative in the formula for the *tangent* of the angle C, &c.

PROBLEM III.

215. *Suppoſing the ſame things given as in the preceding Problem; it is required to find the third ſide without regard to the angles adjacent thereto.*

SOLU-

SOLUTION.

For facility of calculation, we will ſuppoſe the ſides AB and BC with the included angle B to be the parts given: then, it is manifeſt that all we have to do is to find the value of the *ſine* IH, or *coſine* GH, of the ſide AC. Now in order to do this, we ſhall get from the right-angled triangles, GKB, CXO, *coſ.* AB : *ſin.* AB :: CX : XO, or, $b : a :: \frac{cn}{r} : \frac{acn}{br}$; whence, GO or $GX + XO = d + \frac{acn}{br}$: likewiſe, from the right-angled triangles, GKB, GHO, we ſhall get, R : *coſ.* AB :: GO : GH; or, algebraically, $r : b :: d + \frac{acn}{br} : \frac{bdr + acn}{rr} =$ *Coſ.* AC. Q. E. I.

COROLLARY I.

216. Therefore, in general, when in any ſpherical triangle two of the ſides and the included angle are given, we ſhall have the following formulæ for finding the third ſide;

$$Coſ.\,AC = \frac{\pm coſ.\,B \times ſin.\,AB \times ſin.\,BC + R \times coſ.\,AB \times coſ.\,BC}{RR}$$

$$Coſ.\,AB = \frac{\pm coſ.\,C \times ſin.\,AC \times ſin.\,BC + R \times coſ.\,AC \times coſ.\,BC}{RR}$$

$$Coſ.\,BC = \frac{\pm coſ.\,A \times ſin.\,AB \times ſin.\,AC + R \times coſ.\,AB \times coſ.\,AC}{RR}$$

COROLLARY II.

217. From theſe formulæ it will follow that, if V be called the *verſed ſine* of any angle C, we ſhall have, $V = \frac{R^2 \times \overline{coſ.\,\overline{BC - AC} - coſ.\,AB}}{ſin.\,AC \times ſin.\,BC}$. For let R − V be put in the place of *coſ.* C in the expreſſion of *coſ.* AB found in the laſt Corollary, and we ſhall

get, $\cos. AB \times RR = \cos. AC \times \cos. BC \times R + \sin. AC \times \sin. BC \times R - \sin. AC \times \sin. BC \times V$; but, by *art.* 24, $\cos. AC \times \cos. BC + \sin. AC \times \sin. BC = R \times \cos. \overline{BC - AC}$; therefore, by the ſubſtitution of this value and the common methods of reduction, we ſhall find that, $V = \frac{R \times R \cos. \overline{BC - AC} - R \cos. AB}{\sin. AC \times \sin. BC} = \frac{2R \times \cos. \overline{BC - AC} - \cos. AB}{\cos. \overline{BC - AC} - \cos. \overline{BC + AC}}$, by ſubſtituting the value of $\sin. AC \times \sin. BC$, deduced from what hath been demonſtrated in *art.* 26.

COROLLARY III.

218. If, in the numerator of the firſt of the above values of V, we ſubſtitute for $\cos. \overline{BC - AC} - \cos. AB$ what may be deduced from the demonſtration in *art.* 57 to be equal to it, there will ariſe this expreſſion, $V = \frac{2R \times \sin. \frac{1}{2} AB + \frac{1}{2} BC - \frac{1}{2} AC \times \sin. \frac{1}{2} AB + \frac{1}{2} AC - \frac{1}{2} BC}{\sin. AC \times \sin. BC}$; and ſince we have ſhewn, in *art.* 22, that the *verſed-ſine* of an angle $= \frac{2 \sin.^2 \frac{1}{2} \mathit{angle}}{R}$, we ſhall get by this laſt ſubſtitution, $\sin.^2 \frac{1}{2} C = \frac{R^2 \times \sin. \frac{AB + BC - AC}{2} \times \sin. \frac{AB + AC - BC}{2}}{\sin. AC \times \sin. BC}$; and, by extracting the ſquare roots, $\mathit{Sin.} \frac{1}{2} C = \frac{R \times \sqrt{\sin. \frac{AB + BC + AC}{2} - AC \times \sin. \frac{AB + AC + BC}{2} - BC}}{\sqrt{\sin. AC \times \sin. BC}}$ which is the ſame formula with that already given in the ſecond Chapter for finding any angle of a triangle whereof the three ſides are known, and ſhews us the uſe of the algebraical analyſis in the diſcovery of different formulæ.

SCHO-

SCHOLIUM.

219. Supposing still that we knew two of the ſides of a triangle with their included angle, as the ſides, AB, AC, with the included angle A of the triangle BAC, and would find the third ſide BC by means of its ſine, we ſhould get, $c = \frac{\sqrt{b^2f^2q^2+2abfgqr+a^2g^2r^2+p^2f^2r^2}}{r^2}$ *, an expreſſion manifeſtly too complicated to be ever uſed in the buſineſs of calculations.

PROBLEM IV.

220. *Given one ſide and the angles above it; to find* 1°, *either of the other ſides, and* 2°, *the third angle.*

SOLUTION.

Let AB be the ſide given; A, B, the angles above this ſide, and let AC be one of the ſides required; the *ſine* or *coſine* of which muſt conſequently be regarded as unknown. Then we ſhall in the firſt place have, by *art.* 112, *ſ*X or *ſin.* BC $= \frac{pg}{m}$: in the next place, we ſhall get an expreſſion for HO by this analogy, *coſ.* AB : *ſin.* AB : Fig. 17 and 18.

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* Since, by *Prob.* I. $fX^2 - CX^2 = fH^2 - CH^2$, and, by *Prob.* II. $CX = \frac{bfq+agr}{r^2}$, $fX^2 - \frac{b^2f^2q^2+2abfgqr+a^2g^2r^2}{r^4}$ will be $= \frac{f^2r^2-f^2q^2}{r^2} = \frac{p^2\times f^2}{r^2} = \frac{p^2f^2r^2}{r^4}$, conſequently, $c = \frac{\sqrt{b^2f^2q^2+2abfgqr+a^2g^2r^2+p^2f^2r^2}}{r^2}$.

GH : HO, or, $b : a :: g : \frac{ag}{b} =$ HO : we have likewiſe, CH $= \frac{fq}{r}$, and conſequently, CO $=$ HO $\pm$ CH $= \frac{ag}{b} \pm \frac{fq}{r}$: moreover, from the right-angled triangles, GKB, CXO, we ſhall get, R : *coſ.* AB : : CO : CX, or, by ſubſtituting the analytical values of this proportion, $r : b :: \frac{ag}{b} \pm \frac{fq}{r} : \frac{ag}{r} \pm \frac{bfq}{rr} =$ CX : laſtly, ſince the lines, MG, *n*G, *f*X and CX, are proportional, we ſhall have this analogy, R : *coſ.* B : : *ſin.* BC or *f*X : CX, or, algebraically, $r : n :: \frac{fp}{m} : \frac{ag}{r} \pm \frac{bfq}{rr}$; whence we deduce, $\frac{nfp}{m} = ag \pm \frac{bfq}{r}$, and by reduction, $\frac{f}{g} = \frac{arr}{\frac{n}{m}p \mp bq}$, or, by ſubſtituting the values of the letters, *Tang.* AC $=$ $\frac{\text{RR} \times \text{ſin.AB}}{\text{cot.B} \times \text{ſin.A} \mp \text{coſ.AB} \times \text{coſ.A}}$: where the ſign — takes place when the angle A is obtuſe, and the ſign $+$ when it is acute. We might find in like manner that, *Tang.* AB $= \frac{\text{RR} \times \text{ſin.AC}}{\text{cot.C} \times \text{ſin.A} \mp \text{coſ.AC} \times \text{coſ.A}}$, and, *Tang.* BC $= \frac{\text{RR} \times \text{ſin.AB}}{\text{cot.A} \times \text{ſin.B} \mp \text{coſ.AB} \times \text{coſ.B}}$. Q. E. 1°. I.

2°. In order to find the third angle C, after having found either of the ſides, let the equation $\frac{nfp}{m} = ag \pm \frac{bfq}{r}$ be reſumed, and from thence let there be deduced the value of $\frac{am}{f}$; and we ſhall get, $\frac{am}{f} = \frac{rnp \mp bmq}{gr}$; or, *Sin.* C $= \frac{\text{ſin.A} \times \text{coſ.B}}{\text{coſ.AC}} \mp \frac{\text{coſ.AB} \times \text{coſ.A} \times \text{ſin.B}}{\text{R} \times \text{coſ.AC}}$. Q. E. 2° I.

PROBLEM V.

221. *Suppoſing the ſame things given as in the preceding Problem; it is required to find the third angle in expreſſions of the ſides and angles given.*

SOLUTION.

That we may the more eaſily obtain a formula for the part ſought, we will ſuppoſe the two angles A and C with the ſide AC to be known, and the angle B to be that required; the *ſine*, m, or *coſine*, n, of which muſt conſequently be conſidered as unknown. Then we ſhall firſt have, by the common proportion between the *ſines* of angles and thoſe of their oppoſite ſides, *ſin.* AB or BK $= \frac{fb}{m}$, and, *ſin.* BC or fX $= \frac{pf}{m}$: in the next place, we ſhall get from the triangles, GKB, GHO, *coſ.* AB : *ſin.* AB :: GH : HO, or, $\frac{\sqrt{r^2 m^2 - f^2 b^2}}{m} : \frac{fb}{m} :: g : \frac{gfb}{\sqrt{r^2 m^2 - f^2 b^2}}$: we have alſo by conſtruction, R : *coſ.* A :: *ſin.* AC : CH, or, $r : q :: f : \frac{fq}{r} =$ CH; and conſequently, CO $=$ HO $\pm$ CH $= \frac{gfb}{\sqrt{r^2 m^2 - f^2 b^2}} \pm \frac{fq}{r}$: likewiſe, from the right-angled triangles, GKB, CXO, we ſhall get, R : *coſ.* AB. :: CO : CX, or, in ſpecies, $r : \frac{\sqrt{r^2 m^2 - f^2 b^2}}{m} :: \frac{fgb}{\sqrt{r^2 m^2 - f^2 b^2}} \pm \frac{fq}{r} : \frac{f}{mr} \pm \frac{fq.\sqrt{r^2 m^2 - f^2 b^2}}{mrr} =$ CX: moreover, *Fig.* 17 and 18.

ſince, MG, nG, fX and CX, are proportional, we ſhall have, R : *coſ.* B :: fX : CX, or, $r : n :: \frac{pf}{m} : \frac{fgh}{m} \pm \frac{fq\sqrt{rrm^2 - f^2h^2}}{mr}$; and therefore, $\frac{pfn}{m} = \frac{fgh}{m} \pm \frac{fq\sqrt{r^2m^2 - f^2h^2}}{mr}$. Now, if we divide each member of this equation by $\frac{f}{m}$, and take away the radical, we ſhall have $p^2n^2 - 2pghn + g^2h^2 = q^2m^2 - \frac{f^2h^2q^2}{r^2}$; then, if we put $r^2 - n^2$ in the place of m^2, n^2 for $p^2 + q^2$, and divide the whole by r^2, we ſhall get, $n^2 - \frac{2pghn}{r^2} = q^2 - \frac{g^2h^2}{r^2} - \frac{f^2h^2q^2}{r^4}$; which equation, after completing the ſquare according to the common rules for quadratics, and different reductions and ſubſtitutions, becomes, $n^2 - \frac{2pghn}{r^2} + \frac{p^2g^2h^2}{r^4} = \frac{q^2k^2}{r^2}$; whence we deduce, $n = \frac{pgh}{r^2} \pm \frac{qk}{r}$; that is, *Coſ.* B $= \frac{\textit{ſin.}\ A \times \textit{ſin.}\ C \times \textit{coſ.}\ AC}{RR} \pm \frac{\textit{coſ.}\ A \times \textit{coſ.}\ C}{R}$. —— We might find by ſimilar proceſſes that, *Coſ.* C $= \frac{\textit{ſin.}\ A \times \textit{ſin.}\ B \times \textit{coſ.}\ AB}{RR} \pm \frac{\textit{coſ.}\ A \times \textit{coſ.}\ B}{R}$, and that, *Coſ.* A $= \frac{\textit{ſin.}\ B \times \textit{ſin.}\ C \times \textit{coſ.}\ BC}{RR} \pm \frac{\textit{coſ.}\ B \times \textit{coſ.}\ C}{R}$. Q. E. I.

REMARK.

We might have avoided all the calculations which we have made in order to obtain the ſolution of this Problem, by applying the formula in *art.* 213 to a triangle whoſe parts ſhould be ſeverally the ſupplements to thoſe in the triangle BAC, and afterwards deducing from it by proper ſubſtitutions the value of the *coſine* of the angle required; only we judged it better to find this value by a direct proceſs, that we might render the analytical ſolutions as eaſy and familiar as poſſible.

COROLLARY.

222. From the formula, *tang.* AC = $\frac{RR\times \text{ſin. } AB}{\text{cot. } B\times \text{ſin. } A \mp \text{coſ. } AB\times \text{coſ. } A}$ given in *art.* 220, it will be eaſy to find the following values of the *cot.* of this ſide, by inverting the fraction, and putting *cot.* AB for $\frac{\text{coſ. } AB}{\text{ſin. } AB}$; viz. *Cot.* AC = $\frac{\text{cot. } B\times \text{ſin. } A}{\text{ſin. } AB} \mp$ *cot.* AB × *coſ.* A = $\frac{\text{cot. } B\times \text{ſin. } C}{\text{ſin. } BC} \mp \text{cot. } BC\times \text{coſ. } C$.

PROBLEM VI.

223. *Given two ſides and an angle oppoſite to one of them; to find 1°, the angle included between theſe ſides, and 2°, the ſide adjacent to the given angle.*

SOLUTION.

Let the ſides, AB, AC, and the angle B oppoſite to the ſide AC, be ſuppoſed to be given, and let the angle A be that which is required. Then let the equation, $\frac{nfp}{m} = ag \mp \frac{bfq}{r}$, in *art.* 220 be reſumed, and the values of, p, q, or $\frac{p}{q}$ (which are the *ſine*, *coſine* and *tangent* of the angle A) thence deduced, and the thing will be done. Now, if for greater eaſe we put, the *tangent* of the an- Fig. 17.

angle B equal [illegible] and the [illegible] of the side AC = $\frac{f}{g}$ [illegible] be, [illegible] we shall first get, [illegible] by put- [illegible] $\frac{d}{t}$ [illegible] in the place of q [illegible] from whence we deduce this quadratic equation, $p^2 - \frac{2atr^4}{s\times r^4+b^2t^2}\,p = \frac{b^2t^2r^2s^2 - a^2t^2r^4}{s^2\times r^4+b^2t^2}$; and by the common method of calculation, $p = \frac{atr^4+btr\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\times r^4+b^2t^2}$. We might find in like manner, by substituting for p instead of q the value of the *cosine* of this angle, or, $q = \frac{-abt^2r^2+r^3\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\times r^4+b^2t^2}$, and consequently, from these values of p and q, an analytical expression for the *tangent* of this angle. Q. E. 1°. I.

2°. Supposing still that the sides and angle given are AB, AC and B, and that we would find the side BC, or, which will amount to the same, the *sine* f or *cosine* g of this side. If in the Equation, $\frac{pfn}{m} - \frac{bfq}{r} = ag$, before given, we substitute for the unknown quantities, p and q their values, $\frac{cm}{f}$ and $\frac{1}{f}\sqrt{f^2r^2 - c^2m^2}$ (wherein c must be regarded as unknown), we shall get, $cn - ag = \pm\frac{b}{r}\sqrt{f^2r^2-c^2m^2}$; and thence, by the well known rules for solving quadratics, $c = \frac{agnr^2 \pm br\sqrt{b^2f^2m^2+f^2n^2r^2-a^2g^2m^2}}{b^2m^2+n^2r^2}$. We might likewiſe find from the above equation, the value of the *coſine* of BC; but, as the expreſſion, which we ſhould by this means obtain, would be conſiderably too complicated, it will be neceſſary to inquire whether a more ſimple one cannot be obtained by a farther application of Algebra to *fig.* 17. For this end, the rectangled triangles, GKB, GXτ, will give, *coſ.* AB : R :: GX : Gτ, or, $b : r :: d : \frac{dr}{b}$; whence, τH = GH − Gτ = $g - \frac{dr}{b}$: the ſame triangles will likewiſe give, *coſ.* AB : *ſin.* AB :: GX : τX, or, $b : a :: d : \frac{ad}{b}$: alſo, from the right-angled triangles, GKB, CHτ, we ſhall get, *ſin.* AB : R :: τH : Cτ, or, $a : r :: g - \frac{dr}{b} : \frac{gr}{a} - \frac{drr}{ab}$: and conſequently, C$\tau$ + τX or CX $= \frac{gr}{a} - \frac{drr}{ab} + \frac{ad}{b} = \frac{gr - bd}{a}$: moreover, on account of the ordinates to the circle and ellipſis, we ſhall have, GM : Gn :: fX : CX; or, in ſpecies, $r : n :: \sqrt{rr - dd} : \frac{gr - bd}{a}$; from whence we immediately get, $an\sqrt{rr - dd} = grr - bdr$; and therefore, $d = \frac{r^3 bg \pm anr\sqrt{b^2r^2 + a^2n^2 - g^2r^2}}{b^2r^2 + a^2n^2}$. We might find in like manner, by ſolving another quadratic equation, *Sin.* BC or C $= \frac{agnrr \pm br^2\sqrt{a^2n^2 + b^2r^2 - g^2r^2}}{a^2n^2 + b^2r^2}$; and from theſe two values of d and c deduce that of the *tangent* of the ſame ſide likewiſe. Q. E. 2°. I.

PROBLEM VII.

224. *Given two angles, and one of their opposite sides; to find* 1°, *the adjacent side*; 2°, *the third angle, and* 3°, *the third side.*

SOLUTION.

Fig. 17. Let A and B be the two known angles, and the ſide given AC, oppoſite to the angle B, and let the ſide AB be that required. Then, in the equation, $\frac{bfq}{r}+ag=\frac{fnp}{m}$ (which contains all the given parts of the Problem, together with the *ſine*, *a*, and *coſine*, *b*, of the unknown ſide AB), we need only find the value of either of the quantities *a* and *b*, conſidered as unknown, and the thing will be done. But, that we may render the calculation as ſimple as poſſible, we will in the firſt place divide each member of the above equation by *f*; then put *t* for the *tangent*, $\frac{g}{f}$, of the ſide AC; *s* for the *cotangent*, $\frac{n}{m}$, of the angle B, and make the radius equal to unity; and the ſaid equation will become, $bq+at=ps$, or, $q\sqrt{r^2-a^2}=ps-at$; from whence, by raiſing the whole to the ſquare, and ſolving a quadratic, we ſhall eaſily get, *Sin.* AB or $a=\frac{pst+q\sqrt{r^2t^2+q^2r^2-p^2s^2}}{t^2+q^2}$. We might find in like manner, by ſubſtituting for *a*, and ſolving another quadratic equation, *Coſ.* AB or $b=\frac{pqs\pm t\sqrt{r^2t^2+q^2r^2-p^2s^2}}{t^2+q^2}$; and from theſe two values of the ſide AB an algebraical expreſſion

for the *tangent* thereof might alſo be eaſily obtained. Q. E. I.

2°. For the angle C. Since the analogy between the *ſines* of angles and thoſe of their oppoſite ſides gives, *ſin.* C $= \frac{am}{f}$; therefore, in order to obtain an expreſſion for the *ſine* of this angle in given parts of the Problem, it is manifeſt that, we need only multiply the *ſine*, a, of the ſide AB by $\frac{m}{f}$, and there will ariſe, Sin. C $=$

$$\frac{pst \pm q\sqrt{r^2t^2 + q^2r^2 - p^2s^2}}{t^2 + q^2} \times \frac{m}{f}.$$

Q. E. I.

3°. Laſtly, to find the third ſide BC, it is evident that, we need only make uſe of the common analogy, and it will give, *Sin.* BC $= \frac{\textit{ſin.}\ AC \times \textit{ſin.}\ A}{\textit{ſin.}\ B}$. Q. E. I.

PROBLEM VIII.

225. *Given the three ſides of a ſpherical triangle; to find one of its angles.*

SOLUTION.

Let the equation, $CX = \frac{gr - bd}{a}$, found in the ſecond part of the ſixth Problem, be reſumed: then the conſtruction of the figure gives us, $fX : CX :: MG : gG$, or, whence; B $= \frac{gR - bdR}{ac}$; and therefore, by ſubſtituting the value of each letter, *Coſ.* B will be found $=$

$$\frac{\textit{coſ.}\ AC \times RR - \textit{coſ.}\ AB \times \textit{coſ.}\ BC \times R}{\textit{ſin.}\ AB \times \textit{ſin.}\ BC}.$$

We might find in like manner that, *Coſ.* A $=$

$$\frac{\textit{coſ.}\ BC \times RR - \textit{coſ.}\ AB \times \textit{coſ.}\ AC \times R}{\textit{ſin.}\ AB \times \textit{ſin.}\ AC},$$

and, *Coſ.* C $=$

$$\frac{\textit{coſ.}\ AB \times RR - \textit{coſ.}\ AC \times \textit{coſ.}\ BC \times R}{\textit{ſin.}\ AC \times \textit{ſin.}\ BC}.$$

Fig. 171.

PRO-

PROBLEM IX.

226. *Given the three angles of a ſpherical triangle; to find one of its ſides.*

SOLUTION.

By ſtill purſuing the analyſis ariſing from the conſtruction of *fig.* 17. it would be eaſy to obtain an equation containing the *coſine* of the ſide required, expreſt by, or combined with, the given parts of the Problem: but in the triangle DEF, whoſe parts are ſeverally the ſupplements to thoſe in the triangle BAC, and whoſe three ſides are given, we ſhall have, *Coſ.* D $= \frac{cos. EF \times RR - cos. DF \times cos. DE \times R}{sin. DF \times sin. DE}$; and therefore, ſince the angle D is the ſupplement to the ſide AB, and arcs which are the ſupplements to each other have the ſame *ſine* and *coſine*, we ſhall get, by making ſuch ſubſtitutions as the figure requires, *Coſ.* AB $= \frac{cos. C \times RR - cos. A \times cos. B \times R}{sin. A \times sin. B}$. —— We might find in like manner that, *Coſ.* AC $= \frac{cos. B \times RR - cos. A \times cos. C \times R}{sin. A \times sin. C}$, and that, *Coſ.* BC $= \frac{cos. A \times RR - cos. B \times cos. C \times R}{sin. B \times sin. C}$. Q. E. I. *Fig.* 22.

SCHOLIUM I.

227. If the algebraical ſolutions of the ſixth and ſeventh Problems be compared with thoſe which we have demonſtrated ſynthetically in the ſecond Chapter, we ſhall be ſtruck with ſurpriſe at the prodigious difference which will appear between the one and the other; nay, we ſhall be almoſt induced to believe that there is ſome impropriety

in the algebraical analyſis, when we ſee how complicated ſome of the expreſſions are which ariſe from its uſe. This difference, however, deſerves to be examined with the greateſt attention, and may diſcover to us ſome important truths, very uſeful in their applications: (for which reaſon, we ſhall obſerve that, when the triangle BAC is reduced into two other right-angled ones in order to obtain its ſeveral parrs, if we would find the angle A, by ſuppoſing the ſides, AB, AC, and the angle B oppoſite to one of theſe ſides, known; the operations laid down in the general Table for oblique-angled triangles give us the value of this angle by parts; viz. the *cotangent* of BAD in the firſt place, and the *coſine* of CAD in the ſecond; whilſt the algebraical analyſis ſuppoſes no ſuch diviſion of the ſaid angle, but diſcovers its whole *ſine* or *coſine* by one operation: and therefore, from this different manner of proceeding in the ſolution of the Problem, it is, that the difference of the ſolutions ariſeth: which, nevertheleſs, is not real, but only apparent; as the following proceſs will put beyond a poſſibility of doubt. Let x be put for the *ſine* of the angle BAD; then the formula, *cot.* I. ſeg. of the ang. BAC = *tang. given ang.* × *coſ. adj. ſide*, will give, $\frac{r\sqrt{rr-xx}}{x} = \frac{bt}{r}$, by preſerving the ſubſtitutions uſed in *Prob.* VI: from whence we get, x or *ſin.* BAD $= \frac{r^3}{\sqrt{r^4+b^2t^2}}$; and conſequently, *coſ.* BAD $= \frac{btr}{\sqrt{r^4+b^2t^2}}$. The ſecond formula of the ſame article in the ſame Table gives, *coſ.* CAD $= \frac{atrr}{s\sqrt{r^4+b^2t^2}}$; and conſequently, *ſin.* CAD $= \frac{r\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\sqrt{r^4+b^2t^2}}$: Fig. 15.

therefore, by means of theſe values, and ſince BAC = BAD ± CAD, it will be eaſy to find the *ſine* of the whole of this angle from the formula in *art.* 23, $ſin.\ \overline{A \pm B} = \frac{ſin.A \times coſ.B \pm ſin.B \times coſ.A}{R}$ (ſuppoſing BAD = A and CAD = B); viz. $p = \frac{atr^4 \pm btr\sqrt{r^4s^2 + b^2s^2t^2 - a^2t^2r^2}}{s \times \overline{r^4 + b^2t^2}}$; which is exactly the ſame equation with that we found in *Prob.* VI. by ſolving an equation of the ſecond degree. Hence we may perceive, how theſe two kinds of ſolutions correſpond in reality, though apparently quite different; as alſo, with what facility equations of the ſecond degree, which contain very complicated radicals, may be ſolved by two analogies of ſpherical Trigonometry.

SCHOLIUM II. *for right-angled ſpherical Triangles.*

228. But there is yet another difference which may be obſerved to ſubſiſt betwixt the ſynthetic and analytical ſolutions of the Problems of ſpherical Trigonometry: for, the former owe the ſimplicity of their formulæ to this; that we begin with finding values for the moſt ſimple caſes, and afterwards refer to them for the values of the more difficult and complex ones: whereas in the latter, on the contrary, we ſuppoſe the ſolutions of general caſes to be equally ſimple with thoſe of particular ones; and likewiſe, that all particular caſes are contained in the general formulæ, and may be deduced from them with the greateſt facility; as, for inſtance, the ſeveral formulæ which relate to right-angled ſpherical triangles, by conſidering that the *ſine* of a right angle is equal to the radius, but its *coſine* nothing. Thus, from the formula in *art.* 214, $tang.\ B = \frac{ſin.A \times RR}{ſin.AB \times coſ.AC \pm coſ.AB \times coſ.A}$, we ſhall

get, by ſuppoſing the angle A 90°, $\textit{tang.}\,\mathrm{B} = \frac{\textit{tang}\,\mathrm{AC}\times\mathrm{R}}{\textit{ſin.}\,\mathrm{AB}}$: in like manner, the formula in *art.* 216, $\textit{coſ.}\,\mathrm{BC} = \frac{\mathrm{R}\times\textit{coſ.}\,\mathrm{AB}\times\textit{coſ.}\,\mathrm{AC}\pm\textit{coſ.}\,\mathrm{A}\times\textit{ſin.}\,\mathrm{AB}\times\textit{ſin.}\,\mathrm{AC}}{\mathrm{RR}}$, will give, $\textit{coſ.}\,\mathrm{BC} = \frac{\textit{coſ.}\,\mathrm{AB}\times\textit{coſ.}\,\mathrm{AC}}{\mathrm{R}}$. The angle A being ſtill ſuppoſed right, the formulæ in *art.* 220 will give, $\textit{tang.}\,\mathrm{AC} = \frac{\textit{ſin.}\,\mathrm{AB}\times\textit{tang.}\,\mathrm{B}}{\mathrm{R}}$; $\textit{tang.}\,\mathrm{AB} = \frac{\textit{ſin.}\,\mathrm{AC}\times\textit{tang.}\,\mathrm{C}}{\mathrm{R}}$, and, $\textit{tang.}\,\mathrm{BC} = \frac{\textit{tang.}\,\mathrm{AB}\times\mathrm{R}}{\textit{coſ.}\,\mathrm{B}}$; and the formulæ in *art.* 221, $\textit{coſ.}\,\mathrm{B} = \frac{\textit{ſin.}\,\mathrm{C}\times\textit{coſ.}\,\mathrm{AC}}{\mathrm{R}}$, and, $\textit{coſ.}\,\mathrm{C} = \frac{\textit{ſin.}\,\mathrm{B}\times\textit{coſ.}\,\mathrm{AB}}{\mathrm{R}}$. From the value of p found in *art.* 223 we ſhall get, by ſuppoſing the angle B right, $\textit{ſin.}\,\mathrm{A} = \frac{\mathrm{R}\sqrt{\textit{tang.}^2\,\mathrm{AC}-\textit{tang.}^2\,\mathrm{AB}}}{\textit{tang.}\,\mathrm{AC}}$[*]: the formula for the *coſine* of A, upon a ſuppoſition that the angle B is ſtill right, will be immediately reduced to, $q = \frac{ar^2}{sb}$; becauſe the radical in this caſe vaniſhes: therefore, in any ſpherical triangle, right-angled at B, we ſhall have, $\mathrm{R}\times\textit{coſ.}\,\mathrm{A} = \textit{tang.}\,\mathrm{AB}\times\textit{cot.}\,\mathrm{AC}$; and, right-angled at A, $\mathrm{R}\times\textit{coſ.}\,\mathrm{B} = \textit{tang.}\,\mathrm{AB}\times\textit{cot.}\,\mathrm{BC}$. Moreover, from the formula inveſtigated in the ſecond part of the ſame article,

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[*] For if in the equation, $p = \frac{atr^4+btr^4\sqrt{r^4s^2+b^2t^2s^2-a^2t^2r^2}}{s\times\overline{r^4+b^2t^2}}$, the value of t be reſtored, we ſhall eaſily find, $p = \frac{amnr^4+bmr\sqrt{n^2r^4s^2+b^2m^2s^2-a^2m^2r^2}}{n^2sr^4+sb^2m^2}$; which by ſuppoſing $n = 0$ becomes, $p = \frac{bm^2r\sqrt{b^2s^2-a^2r^2}}{sb^2m^2} = \frac{r\sqrt{b^2s^2-a^2r^2}}{bs} = r\sqrt{\frac{b^2s^2-a^2r^2}{s^2b^2}} = r\sqrt{1-\frac{a^2r^2}{s^2b^2}} = \frac{\mathrm{R}\sqrt{\textit{Tang.}\,\mathrm{AC}^2-\textit{Tang.}^2\,\mathrm{AB}}}{\textit{Tang.}\,\mathrm{AC}}$.

article, $d = \frac{r^3bg \pm n\&c.}{b^2r^2 \pm a^2n^2}$, we ſhall eaſily deduce, ſince n is $= 0$, $d = \frac{rg}{b}$, or, $cof.\ BC = \frac{R \times cof.\ AC}{cof.\ AB}$: conſequently, in a triangle right-angled at A, cof. AC will be $= \frac{R \times cof.\ BC}{cof.\ AB}$, and, $fin.\ AC = \frac{R\sqrt{cof.^2 AB - cof.^2 BC}}{cof.\ AB}$. Again, the firſt formula in *art.* 224 gives, $R \times fin.\ AB = tang.\ AC \times cot.\ B$; becauſe the radical, being multiplied by $q = 0$, vaniſhes; but in the ſecond formula, on the contrary, the radical only remains, and we get this equation; $b = \frac{r\sqrt{x^2 - s^2}}{x}$, or, $cof.\ AB = \frac{R\sqrt{cot.^2 AC - cot.^2 B}}{cot.\ AC}$. Laſtly, when we know (beſides the right-angle) the two angles above the hypothenuſe, the formulæ in *art.* 226 will give; $cof.\ AB = \frac{cof.\ C \times R}{fin.\ B}$; $cof.\ AC = \frac{cof.\ B \times R}{fin.\ C}$, and, $cof.\ BC \times R = cot.\ B \times cot.\ C$: and hence we may readily perceive, that it had been eaſy to arrive at the General Theorem of *Neper*, as well by algebraical proceſſes, as ſynthetic conſiderations.——In order to make the generality of the preceding formulæ ſtill farther appear, we might apply them to different Problems in plain Trigonometry; but, as this is a thing very ſimple in itſelf, we ſhall leave it to the exerciſe of Learners, and add, in the next place, ſome conſiderations upon the conſtructions in *art.* 204 and 205.

PROBLEM X.

229. *Suppoſe a perpendicular be let fall from any* Fig. 21. *angle of a ſpherical triangle,* BAC, *upon the oppoſite ſide, produced if neceſſary; it is required to find in ſines, coſines, tangents or cotangents; i. e. the relations of the ſegments*

ſegments of the baſe to their adjacent angles; 2°, *the relations of theſe ſegments to their correſponding ſides*; 3°, *the relations of the ſegments of the vertical angle to their adjacent ſides, and* 4°, *the relations of the ſame ſegments to the angles above the baſe.*

SOLUTION.

Let us ſuppoſe a perpendicular, CP, to be let fall from the point C upon the ſide AB, produced if neceſſary; then, it is manifeſt that, the arcs AP and BP, or the angles ACP and BCP meaſured by theſe arcs, will be the ſegments formed by this perpendicular: it is likewiſe evident that, the half, GC, of the chord GC*g* will be the *ſine* of the ſaid perpendicular, and conſequently GC its *coſine*. This being premiſed, from the right-angled triangle GHC we ſhall firſt get, R : GC or *coſ.* CP :: *ſin.* AP : CH: alſo, ſince the lines, *r*G, *d*G, *l*H and CH, are proportional, we ſhall have, *coſ.* A : R :: CH : *ſin.* AC; conſequently, by multiplying the correſponding terms of theſe two proportions together, we ſhall again have, *coſ.* A : *coſ.* CP :: *ſin.* AP : *ſin.* AC: we might find by a ſimilar proceſs that *coſ.* B : *coſ.* CP :: *ſin.* BP : *ſin.* BC; and therefore that, *Sin.* AP : *ſin.* BP :: *ſin.* AC × *coſ.* A : *ſin.* BC × *coſ.* B :: *ſin.* B × *coſ.* A : *ſin.* A × *coſ.* B (by ſubſtituting *ſin.* B : *ſin.* A for *ſin.* AC : *ſin.* BC) :: $\frac{\textit{ſin.}\ B}{\textit{coſ.}\ B} : \frac{\textit{ſin.}\ A}{\textit{coſ.}\ A}$ :: *tang.* B : *tang.* A :: *cot.* A : *cot* B; or, which is the ſame, that, *The ſines of the ſegments of the baſe are to each other as the cotangents of their adjacent angles.* Q. E. 1°. I. & D.

2°. Since the angle GCH is the complement of the arc AP, and likewiſe the angle GCX the complement of the arc BP, the triangle GHC will give, R : *coſ.* CP :: *coſ.* AP : GH = *coſ.* AC, and the triangle GCX, R : *coſ.* CP :: *coſ.* BP : GX = *coſ.* BC:

conſequently, ſince theſe proportions have their two firſt terms the ſame, *Coſ.* AP *will be* : *coſ.* BP : : *coſ.* AC : *coſ.* BC ; or, *The coſines of the ſegments of the baſe as thoſe of the correſponding ſides.* Q. E. 2°. I. & D.

We might prove in the ſame manner, by letting fall the perpendicular Aδ upon the baſe BC, that, *Sin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C, and, *Coſ.* Bδ : *coſ.* Cδ : : *coſ.* AB : *coſ.* AC.

3°. Suppoſe now that the perpendicular Aδ is in reality let fall, and that the ſegments, BAδ, CAδ, of the angle BAC are actually taken into conſideration : then, as we have already proved, in *art.* 206, that Rθ = Aδ, θζ will of courſe be equal to the *coſine* of this perpendicular. This being granted, on account of the proportional lines, RG, ρG, θζ and Zζ, and ſince ρG is manifeſtly equal to the *coſine* of the angle BAδ, we ſhall have, R : *coſ.* BAδ : : *coſ.* Aδ : ζZ : moreover, ſince ZG is (by *art.* 205) = *ſin.* B, and the angle ZGζ equal to the complement of AB, *coſ.* AB *will be* : R : : Zζ : *ſin.* B ; therefore, if we multiply the correſponding terms of theſe two proportions together, we ſhall get, *coſ.* AB : *coſ.* BAδ, : : *coſ.* Aδ : *ſin.* B : we might find in like manner that, *coſ.* AC : *coſ.* CAδ : : *coſ.* Aδ : *ſin.* C : conſequently, we ſhall alſo have, *Coſ.* BAδ : *coſ.* CAδ : : *ſin.* B × *coſ.* AB : *ſin.* C × *coſ.* AC : : *ſin.* AC × *coſ.* AB : *ſin.* AB × *coſ.* AC : : $\frac{\text{coſ. AB}}{\text{ſin. AB}}$: $\frac{\text{coſ. AC}}{\text{ſin. AC}}$: : *cot.* AB : *cot.* AC ; whence we infer that, *The coſines of the ſegments of the vertical angle are as the cotangents of their adjacent ſides.* Q. E. 3°. I. & D.

4°. From the proportions between the *ſines* of the ſides and thoſe of their oppoſite angles in the right-angled triangles, AδB, AδC, we ſhall get, *ſin.* BAδ

× *ſin.* Aδ = *ſin.* Bδ × *ſin.* B, and, *ſin.* CAδ × *ſin.* Aδ = *ſin.* Cδ × *ſin.* C; and conſequently, *Sin.* BAδ : *ſin.* CAδ : : *ſin.* Bδ × *ſin.* B : *ſin.* Cδ × *ſin.* C : : *coſ.* B : *coſ.* C (becauſe it hath been demonſtrated at the end of the ſecond part of this article that, *ſin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C) : whence it appears that, *The ſines of the ſegments of the vertical angle are to each other as the coſines of the angles above the baſe.* Q. E. 4°. I. & D.

SCHOLIUM.

230. We may now perceive how all the Theorems neceſſary to the reſolution of right or oblique-angled ſpherical triangles may be found, either algebraically or geometrically, by means of the graphical conſtructions explained in the foregoing Chapter. But a little obſervation will ſuffice to convince us, that there are other methods of diſcovering theſe truths beſides thoſe already ſpecified: thus, for inſtance, we might have obtained the laſt analogies from the interſection of the chords, *νδωμ*, *ηδεφ′*; or (if we conſider that the arcs, *pγ*, *pβ*, are the meaſures of the ſegments, ACP, BCP, formed by the perpendicular, CP, with the ſides containing the angle C), from the interſection of the lines, *γλ*, *l′λ*, *βφ*, *fφ*, &c.

If we ſuppoſe the ſpherical triangle BAC to become a right-lined triangle, it will be eaſy to find how the four preceding analogies will be then expreſſed.——The firſt, *ſin.* Bδ : *ſin.* Cδ : : *cot.* B : *cot.* C, becomes, Bδ : Cδ : : *cot.* B : *cot.* C; and therefore in any plain triangle, *The ſegments of the baſe formed by a perpendicular let fall from the vertical angle are as the cotangents of their adjacent angles*: for, if Aδ be regarded as radius, theſe lines will be the *tangents* of the ſegments of the angle BAC, which are the complements of the angles above the baſe.

The ſecond analogy becomes, ∞ : ∞ :: ∞ : ∞. The third gives, *The coſines of the ſegments of the vertical angle in the ſame proportion to each other as are their oppoſite ſides*; becauſe the three angles of any plain triangle, when taken together, are only equal to two right angles. Laſtly, the fourth analogy gives a proportion whereof the alternate terms are identically the ſame, viz. the antecedent equal to the antecedent, and the conſequent to the conſequent.

COROLLARY.

231. If the arc Aδ be ſuppoſed to divide the angle BAC into two equal parts; then, ſince the angles, AδB, AδC, which are the ſupplements to each other, have the ſame *ſine*, and, *ſin.* AδB : *ſin.* AB :: *ſin.* BAδ : *ſin.* Bδ, as alſo, *ſin.* AδC : *ſin.* AC :: *ſin.* CAδ : *ſin.* Cδ; it will follow that in this caſe, *The ſines of the ſegments of the baſe are as the ſines of the ſides oppoſite to them*; and conſequently, in a right-lined triangle, *Theſe ſegments will be to each other as their oppoſite ſides*, whenever the angle included between theſe ſides is biſected.

SCHOLIUM.

232. If we look back upon the ſeveral formulæ which we have diſcovered by the algebraical analyſis, we ſhall perceive that, in order to find the logarithm of any unknown quantity, it is generally requiſite to find that of the ſum or difference of two given quantities. Now we have already ſhewn the method of performing theſe operations, and therefore it will be ſufficient in this place to make application thereof to a few examples.

EXAMPLE I.

Fig. 11. 233. Suppoſe that in the triangle BAC we know the two ſides, AB, BC and the included angle B to be, 41° 9′, 71° 30′ and 27° 8′ reſpectively; it is required to find the angle at A by the formula in *art.* 214, $\mathit{tang.}\,A = \frac{\mathit{ſin.}B \times RR}{\mathit{ſin.}AB \times \mathit{cot.}BC - \mathit{coſ.}AB \times \mathit{coſ.}B}$; where we aſſume the ſign —, becauſe the angle at A is to be acute. Now, it is manifeſt that, the whole difficulty in performing the operation will conſiſt in finding the logarithm of the denominator; but this will be eaſily ſurmounted, by regarding *ſin.* AB × *cot.* BC as one angle, and *coſ.* AB × *coſ.* B as another, and afterwards finding the *ſine* of their difference by the formula in *art.* 92, $\mathit{ſin.}\,A - \mathit{ſin.}\,B = 2\,\mathit{coſ.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \mathit{ſin.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.

OPERATION *by the* LOGARITHMS.

9.818247 = *log. ſin.* AB.
9.524520 = *log. cot.* BC.
9.342767 = *log. ſut.* 12° 43′ 20″ = B.

9.875789 = *log. coſ.* AB.
9.949364 = *log. coſ.* B.
9.826153 = *log. ſin.* 42° 4′ 35″ = A.

9.948331 = *log. coſ.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$.
9.403790 = *log. ſin.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.
0.301030 = *log.* 2.
19.653151 = *log. to be ſubtr. from*
29.659025 = *log. ſin.* B × RR.
10.005874 = *log. tang.* 45° 23′ 14″.

EXAMPLE II.

234. Suppoſe the ſame things given as in the preceding Example; it is required to find the third ſide, AC, by the formula in *art.* 216, $\mathit{coſ.}\,AC = \frac{\mathit{coſ.}B \times \mathit{ſin.}AB \times \mathit{ſin.}BC + \mathit{coſ.}AB \times \mathit{coſ.}BC \times R}{RR}$. —If this quantity be conſidered as the expreſſion of a *ſine* equal to the ſum of the *ſines* of two angles, and the formula in *art.* 91, $\mathit{ſin.}\,A + \mathit{ſin.}\,B = 2\,\mathit{ſin.}\,\overline{\tfrac{1}{2}A + \tfrac{1}{2}B} \times \mathit{coſ.}\,\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$, made uſe of; the operation will be perfectly eaſy, and performed as follows:

9.949364 = *log. coſ.* B.
9.818247 = *log. ſin.* AB.
9.976957 = *log. ſin.* BC.
9.744568 = *log. ſin.* 33° 44′ 5″ = A.
9.875789 = *log. coſ.* AB.
9.591476 = *log. coſ.* BC.
9.378265 = *log. ſin.* 13° 49′ 23″ = B.

$\tfrac{1}{2}A + \tfrac{1}{2}B$ = 23° 46′ 44″.
comp. $\tfrac{1}{2}A - \tfrac{1}{2}B$ = 80° 2′ 39″.

9.605529 = *log. ſin.* $\overline{\tfrac{1}{2}A + \tfrac{1}{2}B}$.
9.993410 = *log. coſ.* $\overline{\tfrac{1}{2}A - \tfrac{1}{2}B}$.
0.301030 = *log. of* 2.
9.899969 = *log. coſ.* 57° 24′ 54″.

EXAMPLE III.

235. Suppoſe now that the three ſides, AB, AC and BC, of the triangle BAC are given, viz. AB = 41° 9′, AC = 62° 35′, and BC = 71° 30′; it is required to find one of the angles of this triangle, as the angle A for inſtance, by the formula in *art.* 225, $\textit{coſ.}\,A = \frac{\textit{coſ.}BC \times RR - \textit{coſ.}AC \times \textit{coſ.}AB \times R}{\textit{ſin.}AB \times \textit{ſin.}AC}$.

The value of this expreſſion will be obtained by means of the formula in *art.* 92, *ſin.* A — *ſin.* B, *&c.* and the operation ſtand thus:

9.783623 = *log. coſ.* AC.
9.876789 = *log. coſ.* AB.
9.660412 = *log. ſin.* 27° 13′ 38″ = A.
coſ. BC = 18° 30′ = B.
therefore, 22° 51′ 49″ = ½ A + ½ B:
and, 4° 21′ 49″ = ½ A — ½ B.
0.181753 = *arith. comp. ſin.* AB.
0.100049 = *arith. comp. ſin.* AC.
0.301030 = *log. of* 2.
9.964463 = *log. coſ.* ½ A + ½ B.
8.881893 = *log. ſin.* ½ A — ½ B.
9.428598 = *log. coſ.* 74° 26′ 15″ = *coſ.* A.

236. Since, in the ſolutions of the ſixth and ſeventh Problems, in *art.* 223 and 224, equations of the ſecond degree have been concerned, it is manifeſt that, there may be certain caſes wherein we may be obliged to ſolve equations of this nature; and as, moreover, all the calculations of practical Aſtronomy are made by the logarithms, it will not be improper to ſhew here the manner of obtaining theſe ſolutions by the Tables of *ſines, &c.* for which reaſon we ſhall annex the two following Problems.

PROBLEM I.

237. *To find by the Tables of ſines, &c. the roots of the equation,* $x^2 \pm 2ax = bb$.

SOLUTION.

From any point, C, with a radius equal to a, let the circle ADBd be deſcribed, and at the extremity of the radius CA the perpendicular AL erected, equal to the quantity b: then, through the centre C and point L, let the line LDCd be drawn, terminated by the circumference at d; and the parts, LD, Ld, will be the roots of the equation. The demonſtration of this is very obvious from the application of Algebra to Geometry; for the *tangent* AL and *ſecant* Ld give, Ld : LA :: LA : LD, or, algebraically, $a + \sqrt{aa + bb} : b :: b :$ LD (x); whence we deduce, $ax + x\sqrt{aa + bb} = bb$, or, $x\sqrt{aa + bb} = bb - ax$; and conſequently, by ſquaring both ſides of the equation, ſtriking out the terms that deſtroy, and dividing the remainder by bb, $x^2 + 2ax = bb$. *Fig.* 24.

Now, in order to reduce this conſtruction to the logarithms of the Tables, let there be again erected the right line FLf perpendicular to the extremity of AL, and terminated at, F, f, by the prolongations of the chords, AD, Ad; and let the chords, BD, Bd, be drawn. This being perfectly underſtood, it is eaſy to perceive that, in the right-angled triangle CAL, the angle ACL will, in the firſt place, be determined by this analogy; CA (a) : AL (b) :: R : *tang.* ACL; then, on account of the ſimilar triangles, BDA, ALF, we ſhall have, BD : DA :: AL : LF* = LD (x); but the angle ABD is evidently the half of the angle ACL before found; and therefore we ſhall laſtly have, R : [illegible] *tang.*

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* The triangles ADB and ADd are manifeſtly equal to each other, and alſo ſimilar to the triangle ALF; whence, the angle ADd = the angle FDL = the angle AFL, and conſequently, LF = LD.

tang. ½ ACL :: AL (*b*) : LF or DD (*x*). Conſequently, from theſe proportions the value of *x* will be obtained by the logarithms. Q. E. 1°. I.

2°. If L*d* be called *x* we ſhall find from the ſame proportion (L*d* : LA :: LA : LD) that, $x^2 - 2ax = bb$. Now, to obtain the value of *x* in this caſe, we muſt firſt ſay, as before, *a* : *b* :: *r* : *tang.* ACL; then, on account of the ſimilar right-angled triangles, B*d*A, AL*f*, we ſhall have, B*d* : *d*A :: AL : L*f* = L*d*; but, B*d* : *d*A :: R : *cot. half the angle* BC*d* = ACL; therefore, we ſhall laſtly have, *as* R : *cot. of half the angle found by the laſt proportion* :: *the quantity b* : *to the line* L*f*; which will be the poſitive root of the equation, $xx - 2ax = bb$. Q. E. 2°. I.

PROBLEM II.

1038. *To find by the Tables of ſines, &c. the roots of the equation,* $xx \mp 2ax + bb = 0$.

SOLUTION.

Fig. 25. At the extremity of the radius CA = *a* let there be erected a perpendicular AL = *b*; then through L let there be drawn the line LF*f* parallel to AB (which will cut the circle deſcribed upon AB in two points F and *f*, or otherwiſe touch it in only one point, if the Problem be poſſible), and LF, L*f* will be roots of the equation, $x^2 \mp 2ax + bb = 0$. For, calling FL, *x*; on account of the *tangent* AL and *ſecant* L*f*, we ſhall have, L*f* $(a + \sqrt{aa - bb})$: AL (*b*) :: AL (*b*) : LF (*x*); from whence we ſhall eaſily get, by taking the product of the extremes and means, $x^2 + 2ax + bb = 0$. We might have found in like manner, by calling L*f*, *x*, $x^2 - 2ax + bb = 0$. This being premiſed, if we again draw the lines, AF, A*f*, FB, FC, FD and *fd*, we ſhall, in the firſt place, get from the right-angled

triangle CDF, CF : FD or $a : b :: r :$ *ſine of the angle* FCD : then, on account of the ſimilar triangles, BFA, FDA, we ſhall have, BF : FA :: FD : DA; but the angle at B is manifeſtly the half of the angle ACF at the centre, and therefore we ſhall have, BF : FA :: R : *tang.* ½ FCA : conſequently, we ſhall laſtly have, R : *tang.* ½ FCA :: FD (b) : AD (x); from whence it follows that AD or x will become known, ſince the three firſt terms of this proportion are known.

2°. If Lf be called x, we ſhall in this caſe have, on account of the ſimilar triangles, AFB, fdA; AF : BF :: fd : dA = Lf, or, r : *cot.* ½ FCA :: $b : x$. Q. E. I.

We ſometimes find in Aſtronomy ſuch Problems as lead us into equations of the third degree, but particularly in the calculations of the motions of Comets in a parabolical orbit; for which reaſon we hope our readers will not be diſpleaſed to ſee here the method of ſolving equations of *this* kind likewiſe by the Tables of *ſines*, &c. But before we do this, we ſhall premiſe in a few words what relates to theſe equations in general; and refer, for the demonſtration of the different articles, to what Mr. *Clairaut* hath written upon this ſubject in his Elements of Algebra.

We will ſuppoſe then as truths demonſtrated in ſeveral Treatiſes of Algebra:

1°. That every equation of the third degree may be reduced to the form, $x^3 \pm px \pm q = 0$; wherein the ſecond term is taken away.

2°. That of the roots of the equation, $x^3 + px \pm q = 0$, there are neceſſarily two imaginary and one real.

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* For theſe two Problems ſee *Simpſon*'s Trigonometry. p. 64, &c. 2d edition.

3°. That in the equation, $x^3 - px \pm q = 0$, there are neceſſarily two imaginary roots, whenever $\frac{1}{27}p^3$ is leſs than $\frac{1}{4}qq$.

4°. That, on the contrary, the ſame equation will have its three roots real, when $\frac{1}{27}p^3$ (always negative) is greater than $\frac{1}{4}qq$; which conſtitutes the caſe called the *irreducible* caſe.

Now though theſe conſiderations are not abſolutely neceſſary to the underſtanding of the ſubſequent ſolutions, which are in a great meaſure independent thereon; yet they will ſerve to convince us that theſe ſolutions contain all the poſſible caſes of cubic equations, after the ſecond term is taken away, according to the common and well known methods of transformations.

PROBLEM I.

240. *To find the three roots of an equation of the third degree of the form,* $x^3 - px \pm q = 0$, *by ſuppoſing* $\frac{1}{27}p^3$ *greater than* $\frac{1}{4}qq$; that is to ſay, *in the irreducible caſe.*

SOLUTION.

Let the equation, $4x^3 - 3r^2x \pm r^2y = 0$, be aſſumed (which by means of the indeterminate quantity y may repreſent all cubic equations belonging to the irreducible caſe), and let r be ſuppoſed therein to be greater than y, in order to have $\frac{1}{27}p^3$ greater than $\frac{1}{4}qq$: then, if the laſt term hath the ſine + prefixed to it, put $y = ſin.\ A$, and the circumference of a circle $= C$, and the roots of the equation will be, $x = ſin.\ \frac{A}{3}$; $x = ſin.\ \frac{A+C}{3}$, and, $x = ſin.\ \frac{A+2C}{3}$.
——If the laſt term hath the ſign − prefixed to it, let y be put $= coſ.\ A$, and the roots of the equation ($4x^3 - 3r^2x - r^2y = 0$) will be, $x = coſ.\ \frac{A}{3}$; $x =$

coſ. $\frac{A+C}{3}$, and, $x = coſ. \frac{A+2C}{3}$. The demonſtration of theſe operations is very eaſily deduced from the general formulæ in *art.* 53. & 54, concerning the *ſines* and *coſines* of multiple arcs. Q. E. I.

PROBLEM II.

241. *To find the real root of the equation,* $x^3 - px \mp q = 0$, *when* $\frac{1}{27}p^3$ *is ſuppoſed to be leſs than* $\frac{1}{4}qq$; *the other two roots being, in that caſe, imaginary.*

SOLUTION.

Let the ſame equation ($4x^3 - 3r^2x \mp r^2y = 0$) be aſſumed, as before; in which let r be ſuppoſed leſs than y, that $\frac{1}{27}p^3$ may be leſs than $\frac{1}{4}qq$: then, if the laſt term be ſuppoſed negative, put $y = coſec. 2A$ (which will neceſſarily become known when y is determined); after which find an angle, B, whoſe *cot.*³ may $= r^2$ *cot.* A, and the value of x will be *coſec.* 2B. —— If the laſt term be affirmative, we muſt then conſider y as the *coſecant* of an obtuſe angle, and the ſolution of the Problem will be obtained in the ſame manner. Q. E. I.

PROBLEM III.

242. *To find the real root of the equation,* $x^3 + px \mp q = 0$; *the other two roots being imaginary, ſince* $\frac{1}{27}p^3$ *will neceſſarily be poſitive.*

SOLUTION.

Let the ſame equation ($4x^3 + 3r^2x \mp r^2y = 0$) be ſtill retained; in which make $y = \pm$ *cot.* 2A, and afterwards find an angle, B, ſo that *cot.*³ B may $= \pm r^2$ *cot.* A; and the value of the unknown quantity x will be $\pm$ *cot.* 2B. Q. E. I.

DEMONSTRATION *of the two laſt Solutions.*

243. Whether r be greater or leſs than y, it may always be truly affirmed that the equation, $4x^3 -$

$3r^2x - r^2y = 0$, is the ſame with, $\overline{x+\sqrt{x^2-r^2}}\big|^3 = r^2 \times \overline{y+\sqrt{y^2-r^2}}$; and conſequently that the value of x in y obtained from the latter, will ſolve the former equation. All that is requiſite to be done then is to prove the identity of theſe two expreſſions. And in order to this, we ſhall in the firſt place obſerve, that by the formula in *art.* 29 we in reality have, $4x^3 - 3r^2x - r^2y = 0$: ſo much granted, let it, in the next place, be allowed as a ſuppoſition that, $\overline{x+\sqrt{x^2-r^2}}\big|^3 = r^2 \times \overline{y+\sqrt{y^2-r^2}}$; then by expanding the cube we ſhall get, $4x^3 - 3r^2x - r^2y + \overline{4x^2 - r^2} \times \sqrt{x^2-r^2} = r^2\sqrt{y^2-r^2}$; but herein the terms, $4x^3 - 3r^2x - r^2y$, occur, which have been already put $= 0$, and therefore it remains only to be proved that, $\overline{4x^2 - r^2} \times \sqrt{x^2-r^2} = r^2\sqrt{y^2-r^2}$. Now let both ſides of this equation be raiſed to the ſquare, and we ſhall find, $16x^6 - 24r^2x^4 + 9r^4x^2 - r^6 = r^4y^2 - r^6$, or, $16x^6 - 24r^2x^4 + 9r^4x^2 = r^4y^2$; which is exactly the ſquare of the equation, $4x^3 - 3r^2x = r^2y$; and therefore the truth of what was required to be proved follows of courſe.

If now as in *Prob.* II. we ſuppoſe r to be leſs than y, and put $y = \textit{coſec.}\ 2A$, we ſhall get, $\sqrt{y^2 - r^2} = \textit{cot.}\ 2A$: likewiſe, if $x = \textit{coſec.}\ 2B$, $\sqrt{x^2-r^2}$ will be $= \textit{cot.}\ 2B$: therefore, by ſubſtituting theſe values in the equation $\overline{x+\sqrt{x^2-r^2}}\big|^3 = r^2 \times \overline{y+\sqrt{y^2-r^2}}$, we ſhall get, $\overline{\textit{coſec.}\ 2B + \textit{cot.}\ 2B}\big|^3 = r^2 \times \overline{\textit{coſec.}\ 2A + \textit{cot.}\ 2A}$; and ſince, by the formula in *art.* 84, we have, $\textit{coſec.}\ 2A + \textit{cot.}\ 2A = \textit{cot.}\ A$, the laſt equation, after proper ſubſtitution in each member thereof, will become, $\textit{cot.}^3\ B = r^2\ \textit{cot.}\ A$: from whence it will be eaſy to find *cot.* B, when A is known; and conſequently the value of $x =$ by ſuppoſition to *coſec.* 2B. Q. E. 1°. D.

2°. We may prove in the ſame manner that the equation, $4x^3 + 3r^2x - r^2y = 0$, is identically the ſame with, $\overline{x + \sqrt{x^2 + r^2}}|^3 = r^2 \times \overline{y + \sqrt{y^2 + r^2}}$: but this equation, by ſuppoſing as in *Prob.* III. $y = cot. 2A$, and $x = cot. 2B$, will become, $\overline{cot. 2B + coſec. 2B}|^3 = r^2 \times \overline{cot. 2A + coſec. 2A}$; and, in the laſt place, $cot.^3 B = r^2 cot. A$: whence it will be eaſy to find the arc denoted by B, when the value of A is known; and conſequently *cot.* 2B, which is the value of the unknown quantity x in the equation. Q. E. 2°. D.

We ſhall now, in order to make the utility and nature of theſe ſolutions more clearly appear, apply them to particular examples.

EXAMPLE I.

244. *To find the roots of the equation,* $x^3 - 3x - 1 = 0$; *which will be all three real, becauſe* $\frac{1}{27} p^3$ *is negative, and greater than* $\frac{1}{4} qq$.

SOLUTION.

That this equation may be compared with $4x^3 - 3r^2x - r^2y = 0$, I firſt divide all the terms of the latter by 4, and find, $x^3 - \frac{3}{4} r^2 x - \frac{1}{4} r^2 y = 0$: then will $\frac{3}{4} r^2 = 3$; whence $r = 2$, and conſequently $y = 1$. —— Now becauſe the laſt term is negative, y, according to what has been ſaid in *Prob.* I. *art.* 240, muſt be put $=$ *coſ.* A; and ſince this *coſine* is half the ſine total, it muſt neceſſarily be the *ſine* of an angle of 30°, and therefore A itſelf $= 60°$. Conſequently, the values of x will be, *coſ.* 20°; *coſ.* 140°, and, *coſ.* 260°; or, *ſin.* 70°; + *ſin.* 50°; and, − *ſin.* 10°, to a radius expreſſed by 2; that is to ſay, the real values of x will be double the *ſines* corresponding to theſe angles in the Tables.

EXAMPLE II.

245. *To find the real root of the equation,* $x^3 - x - 6 = 0$; *the other two being neceſſarily imaginary, ſince* $\frac{1}{27} p^3$ *is leſs than* $\frac{1}{4} qq$.

SOLU-

SOLUTION.

This equation I compare with, $4x^3 - 3r^2x - r^2y = 0$, after dividing all the terms of the latter by 4, and find $\frac{3}{4}r^2 = 1$; whence $r = \frac{2}{\sqrt{3}}$, and $y = 18$, which must be put = *cosec.* 2A. Then, to obtain this angle by the Tables, I in the first place say, $\frac{2}{\sqrt{3}} : 18 ::$ *sine total of the Tables* (1) : $9\sqrt{3}$ = *cosec.* 2A: but, by *art.* 81, *cosec.* A = $\frac{rr}{\text{sin. A}}$, and therefore, *sin.* 2A = $\frac{rr}{\text{cosec. 2A}}$: whence, by subtracting from double the logarithm of the sine total, the logarithm of $9\sqrt{3}$, *sin.* 2A will be found = 0.807196, or, according to the index in the Tables, 8.807196 = 3° 40′ 40″; and consequently, A = 1° 50′ 20″. ——In the next place, to find the angle B, which ought to determine the value of x by the *cosecant* of its double, to the number 11.49347 let twice the logarithm of the sine total be added, and a third of the resulting sum taken, and the quotient will give, *tan.* B = *cot.* 17° 37′ 50″, or, B = 17° 37′ 50″, the *cosecant* of the double whereof, that is, the *secant* of 54° 44′ 20″, multiplied by $\frac{2}{\sqrt{3}}$ will be the root required of the proposed equation.

EXAMPLE III.

246. *To find the real root of the equation,* $25x^3 + 75x - 46x = 0$; *the other two being necessarily imaginary, because the second term is affirmative.*

SOLUTION.

That this equation may be compared with, $4x^3 + 3r^2x - r^2y = 0$, I first divide all the terms of the latter by 4, and thoſe of the propoſed one by 25; after which I deduce, from a compariſon of the correſponding terms of the two reſulting equations ($x^3 + \frac{3}{4}r^2x - \frac{1}{4}r^3y = 0$, and, $x^3 + 3x - \frac{46}{25} = 0$), $r = 2$, and $y = \frac{46}{25}$. This done, in order to find what the angle A muſt be, ſo that $\frac{46}{25}$ may be the *cotangent* of its double, I make this proportion; *radius (before found 2)* : $\frac{46}{25}$ :: *ſine total of the Tables* (1.000000) : *tan.* 2 A *in Prob.* III; but, by *art.* 12, *tang.* $A = \frac{rr}{cot. A}$; or, *tang.* $2A = \frac{rr}{cot. 2A}$; and therefore, by performing the neceſſary logarithmical operation, *tang.* 2A will be found = 10.036212 = 47° 23′ 10″, and conſequently, A = 23° 41′ 35″. The value of A being thus obtained, that of B may be found as in the preceding Problem to be = 37° 13′ 54″; and if the *cotangent* of twice this angle be multiplied by 2 (becauſe $r = 2$, as above ſaid), the product will be the root of the equation. Q. E. I.

OBSERVATION.

It may be obſerved, that theſe ſolutions of equations of the third degree by the circle neceſſarily contain thoſe of equations of the fourth degree likewiſe; ſince it is demonſtrated in moſt Treatiſes of Algebra, that equations+ of this degree may be reduced to thoſe of the third. It may likewiſe be obſerved, that it would be very eaſy to find particular equations in any degree which, as well as the foregoing ones, would admit of ſolutions by means of the circle; but we think it unneceſſary to ſpecify inſtances, as what we have already ſaid will be ſufficient for common practice.

+ See Clairault's Algebra p. 303, or Martin's M. p. 273. or my Algebra p. 26

CHAP. V.

Containing the fluxional Analogies of spherical and plain Triangles.

SECT. I.

LEMMA I.

247. *The fluxion of an arc is to that of its ſine, as radius is to the coſine of this arc.*

DEMONSTRATION.

Fig. 26. LET there be an arc, AM, which we ſhall call z; of which let the *ſine*, MP, be denoted by s; the *coſine*, CP, by u; the *tangent*, AT, by t, and the *ſecant*, CT, by y. Then we muſt prove that, $\dot{z} : \dot{s} :: r : u$.

Now, in order to this, let us ſuppoſe the arc AM to become Am, and let the lines, Cmt, mp, be drawn; alſo, from the centre C with the radius CT, let the arc TS be deſcribed, and through the point M the line Mr drawn parallel to CA: then it is manifeſt by this conſtruction that, Mm will be $= \dot{z}$; $mr = \dot{s}$; M$r = -\dot{u}$ (becauſe the arc AM increaſing, its *coſine* decreaſes; T$t = \dot{t}$, and, tS $= \dot{y}$. So much being premiſed, on account of the ſimilar triangles, mrM, CPM, we ſhall have, CM (r) : CP (u) :: Mm $(\dot{z})$: mr $(\dot{s})$. Q. E. D.

LEMMA II.

248. *The fluxion of an arc is to that of its tangent, as radius ſquared is to the ſquare of the ſecant*; that is, $\dot{z} : \dot{t} :: rr : yy$.

DEMONSTRATION.

Fig. 26. On account of the ſimilar ſectors, CMm, CTS, we ſhall have, CM (r) : CT (y) :: Mm $(\dot{z})$: TS =

$y\dot{z}$

$\frac{y\dot{z}}{r}$: likewiſe, on account of the right-angled triangles, CAT, T*S*t, which are alſo ſimilar (ſince, when the lines, CT, C*t*, are indefinitely near to a coincidence, the angle at *t* will not eſſentially differ from that at T), we ſhall have, CA : CT :: ST : T*t*; or, $r : y :: \frac{y\dot{z}}{r} : \dot{t}$; whence we immediately deduce, $\dot{z} : \dot{t} :: r^2 : y^2$. Q. E. D.

LEMMA III.

249. *The fluxion of an arc is to that of its ſecant, as radius ſquared is to the rectangle under the ſecant and tangent; that is,* $\dot{z} : \dot{y} :: r^2 : ty$.

DEMONSTRATION.

It is evident that S*t* is the fluxion of the *ſecant*, and ST hath already been found to be $\frac{y\dot{z}}{r}$, therefore, by again comparing the homologous ſides of the ſimilar triangles, CAT, TS*t*, we ſhall get, CA : AT :: ST : S*t*; or, $r : t :: \frac{y\dot{z}}{r} : \dot{y}$; and conſequently, $\dot{z} : \dot{y} : r^2 :: ty$. Q. E. D.

COROLLARY.

250. Hence, by collecting theſe different expreſſions, we ſhall have for the fluxion of any arc z; $\dot{z} = \frac{r\dot{s}}{u} = -\frac{r\dot{u}}{s}$ (by making uſe of the fluxion of the *coſine*) $= \frac{rr\dot{t}}{yy} =$ (by *art.* 15) $\frac{r^2\dot{y}}{ty} =$ (by *art.* 11) $\frac{ru\dot{y}}{sy}$.——We might proceed to find with equal facility other formulæ for the *cotangents* and *coſecants*; but thoſe which we have already found will be ſufficient for all that we ſhall have to demonſtrate in this Chapter.

SCHO-

SCHOLIUM.

251. We might have demonſtrated the ſeveral preceding analogies by the formulæ obtained in the firſt Chapter. For inſtance, the firſt Lemma might have been immediately deduced from *art.* 25. $\text{ſin.}\ \overline{A+B} = \text{ſin.}\ A \times \text{coſ.}\ B + \text{coſ.}\ A \times \text{ſin.}\ B$: for, $\dot{\overline{\text{ſin. AM}}} = \text{ſin.}\ \overline{AM + mM} - \text{ſin. AM}$; but, in an evaneſcent arc, z, the *coſine* is equal to radius or unity, and the *ſine* coincides with the arc itſelf; wherefore we ſhall have, $\text{ſin.}\ \overline{x+z} = \text{ſin.}\ z + \frac{\dot{z}\ \text{coſ.}\ z}{r} = s + \frac{v\dot{z}}{r}$, and conſequently, $\dot{s}$ or $\text{ſin.}\ \overline{z+\dot{z}} - s = s + \frac{v\dot{z}}{r} - s = \frac{v\dot{z}}{r}$. The other formulæ might have been as eaſily deduced from the formulæ inveſtigated in the ſame Chapter.

LEMMA IV.

252. *If from the three angles of any ſpherical triangle,* BAC, *as poles, there be deſcribed upon the ſurface of the ſphere another ſpherical triangle,* DEF; *the fluxions of the angles,* D, E, F, *of this triangle will be reſpectively equal to thoſe of the oppoſite ſides,* AB, AC, BC, *of the triangle* BAC, *and the fluxions of the ſides,* DE, EF *and* FD, *of the ſaid triangle the ſame as thoſe of the oppoſite angles in the triangle* BAC; *and contrarily, for the fluxions of the parts of the triangle* BAC *with reſpect to the parts of the triangle* DEF. Fig. 11.

DEMONSTRATION.

This propoſition is an evident conſequence of what hath been demonſtrated in *art.* 190. and of this; that angles, which are the ſupplements to each other, have neceſſarily the ſame *ſine*, *coſine*, *tangent*, *cotangent*, *ſecant* and *coſecant*. Q. E. D.

SECT.

SECT. II.

Concerning the fluxions of any ſpherical or plain triangle, wherein an angle and one of its adjacent ſides are ſuppoſed conſtant.

THEOREM I.

253. *If in any ſpherical triangle,* BAC, *we ſuppoſe an angle,* A, *conſtant, together with one of the ſides,* AC, *adjacent to this angle, we ſhall always have this analogy; as the fluxion of the other ſide adjacent to the conſtant angle is to that of the ſide oppoſite, ſo is radius to the coſine of the angle oppoſite to the conſtant ſide;* *Fig.* 27.
that is, $\dot{\overline{AB}} : \dot{\overline{BC}} :: R : cof.\, B$.

DEMONSTRATION.

Suppoſing the ſide AB to flow into Aβ, or to be increaſed by the indefinitely ſmall quantity Bβ; if Cβ be drawn, and C*b* taken thereon = CB, by deſcribing from the pole C the little circular arc B*b*; *b*β will be the fluxion of the ſide CB, and the little triangle B*b*β will be right-angled at *b*, as alſo (becauſe on account of the ſmallneſs of its ſides it may be conſidered as right-lined) have its ſides proportional to the *ſines* of their oppoſite angles: but the angle at β is eſſentially equal to the angle B, and conſequently the angle βB*b* the complement of the ſame angle; wherefore we ſhall have, Bβ : *b*β : R : *cof.* B, or, $\dot{\overline{AB}} : \dot{\overline{BC}} :: R : cof.\, B$. Q. E. D.

COROLLARY I.

254. From this proportion, and the formulæ demonſtrated in the preceding Chapter, the following analogies may be eaſily deduced; $\dot{\overline{AB}} : \dot{\overline{BC}} ::$ *ſin.* AB × *ſin.* BC : R × *coſ.* AC − *coſ.* AB × *coſ.* BC (by ſubſtituting for *coſ.* B its value in *art.* 225) :: $R : \frac{cof.\,AC \times fin.\,A \times fin.\,C}{RR} - \frac{cof.\,A \times cof.\,C}{R}$ (by *art.* 221)
:: RR

:: RR : *coſ.* AC × *ſin.* C (if the angle A be 90°) or *tang.* BC : *tang.* AB, by *art.* 228.

COROLLARY II.

255. If the triangle is right-lined, we ſhall ſtill have under the ſame ſuppoſitions, $\dot{\overline{AB}} : \dot{\overline{BC}} :: R :$ *coſ.* B.

THEOREM II.

256. *If the ſame things be ſuppoſed as in the preceding Theorem; the fluxion of the variable ſide adjacent to the conſtant angle is to that of the angle oppoſite to this ſide, as the ſine of the ſide oppoſite to the conſtant angle is to the ſine of the angle oppoſite to the conſtant ſide*; that is, $\dot{\overline{AB}} : \dot{C} ::$ *ſin.* BC : *ſin.* B.

DEMONSTRATION.

Let the ſides, CB, CA, be produced till they be each 90°; and, it is manifeſt that, the arc F*f* will be the meaſure of the variation of the angle C. This being granted, ſince the little arcs, F*f*, B*b*, contain the ſame number of degrees, we ſhall have, F*f* : B*b* :: R : *ſin.* BC; and likewiſe, on account of the triangle B*b*β, B*b* : Bβ :: *ſin.* B : R: then, if we multiply the correſponding terms of theſe two proportions together, we ſhall get, F*f* : Bβ :: *ſin.* B : *ſin.* BC; or, $\dot{C} : \dot{\overline{AB}} ::$ *ſin.* B : *ſin.* BC, and, *invertendo*, $\dot{\overline{AB}} : \dot{C} ::$ *ſin.* BC : *ſin.* B. Q.E.D.

COROLLARY I.

257. If for the quantities, *ſin.* B and *ſin.* BC, there be ſubſtituted their different values deduced from the proportion between the *ſines* of angles and thoſe of their oppoſite ſides, we ſhall find this ſeries of equal compariſons; $\dot{\overline{AB}} : \dot{C} ::$ *ſin.*^{2} BC : *ſin.* AC × *ſin.* A :: *ſin.* AC × *ſin.* A : *ſin.*^{2} B :: *ſin.* BC × *ſin.* AB : *ſin.* C × *ſin.* AC :: *ſin.* A × *ſin.*^{2} AB : *ſin.* AC × *ſin.*^{2} C.

COROL-

COROLLARY II.

258. If the triangle be right-lined, the *ſine* of the ſide BC will then become the ſide itſelf, and therefore the Theorem expreſſed thus; $\dot{\overline{AB}}$: $\dot{C}$ :: BC : *ſin.* B.

THEOREM III.

259. *The fluxion of the ſide oppoſite to the conſtant angle is to that of the variable angle adjacent to the conſtant ſide, as the ſine of this ſide multiplied by the coſine of the third angle is to the ſine of the ſame angle multiplied by the radius*; that is, $\dot{\overline{BC}}$: $\dot{C}$ :: *ſin.* BC × *coſ.* B : *ſin.* B × R :: *ſin.* BC : *tang.* B.

DEMONSTRATION.

Since we have, by *art.* 253, $\dot{\overline{BC}}$: $\dot{\overline{AB}}$ :: *coſ.* B : R; and alſo, by the laſt Theorem, $\dot{\overline{AB}}$: $\dot{C}$ :: *ſin.* BC : *ſin.* B; therefore we ſhall get, by multiplying the correſponding terms of theſe two proportions together, $\dot{\overline{BC}}$: $\dot{C}$ :: *ſin.* BC × *coſ.* B : *ſin.* B × R :: *ſin.* BC : *tang.* B. Q. E. D.

COROLLARY I.

260. If we ſubſtitute for *ſin.* BC its value, $\frac{\text{ſin. AC} \times \text{ſin. A}}{\text{ſin. B}}$, this new analogy will ariſe, $\dot{\overline{BC}}$: $\dot{C}$ :: *ſin.* A × *ſin.* AC : *tang.* B × *ſin.* B :: *ſin.*^{2} BC × *cot.* AC — *ſin.* BC × *coſ.* BC × *coſ.* C : R^2 × *ſin.* C, by *art.* 214.

COROLLARY II.

261. If the triangle be right-lined, the ſame ſuppoſitions will give, $\dot{\overline{BC}}$: $\dot{C}$ :: BC : *tang.* B; for the *ſine* of BC then becomes the ſide BC itſelf.

THEOREM IV.

262. *Let the ſame things be ſtill ſuppoſed, and I ſay that; the fluxion of the ſide adjacent to the conſtant angle*

angle is to that of the angle adjacent to this ſide, as the tangent of the ſide oppoſite to the conſtant angle is to the ſine of the angle oppoſite to the conſtant ſide; that is, $\dot{\overline{AB}} : \dot{B} ::$ *tang.* BC : *ſin.* B.

DEMONSTRATION.

According to the ſuppoſitions of the Theorem, *Fig.* 11. in the triangle DEF, whoſe parts are all, by conſtruction, ſupplements to thoſe of the triangle BAC, the angle at E and the ſide DE, adjacent thereto, will be conſtant. Moreover, by the fourth Lemma of this Chapter, the fluxions of the angle D and ſide DF will be reſpectively equal to thoſe of the ſide AB and angle B of the triangle BAC: but, by the laſt Theorem, we have, $\dot{\overline{DF}} : \dot{D} ::$ *ſin.* DF : *tang.* F; and therefore, ſince angles ſupplemental to each other have the ſame *ſine* and *tangent*, we ſhall get, by ſubſtituting for theſe quantities their correſponding values in the triangle BAC, and, *invertendo*, $\dot{\overline{AB}} : \dot{B} ::$ *tang.* BC : *ſin.* B. Q. E. D.

COROLLARY I.

263. Since we have, $\dot{\overline{AB}} : \dot{B} ::$ *tang.* BC : *ſin.* B, and alſo, by *art.* 256, $\dot{C} : \dot{\overline{AB}} ::$ *ſin.* B : *ſin.* BC; we ſhall find, by multiplying theſe two proportions together, $\dot{B} : \dot{C} ::$ *ſin.* BC : *tang.* BC :: *coſ.* BC : R (*art.* 11) :: *coſ.* A × R — *coſ.* B × *coſ.* C : *ſin.* B × *ſin.* C (*art.* 226) :: *coſ.* A × *ſin.* AC × *ſin.* AB + *coſ.* AC × *ſin.* AB × R : R^2, by putting for *coſ.* BC its value in *art.* 216.

COROLLARY II.

264. If the triangle be right-lined, *ſin.* BC will then be = *tang.* BC, and therefore the fluxions of the angles B and C alſo equal: which thing muſt of neceſſity happen; ſince in every right-lined tri-

angle the ſum of the three angles is always a conſtant quantity, and moreover by ſuppoſition one of the angles, A, conſtant.

SECT. III.

Concerning the fluxions of any ſpherical or right-lined triangle, wherein one of the angles and its oppoſite ſide are ſuppoſed conſtant.

THEOREM V.

265. *Suppoſe that in the triangle* BAC *the angle at* A *and its oppoſite ſide,* BC, *are conſtant; the fluxions of either of the other angles will be to that of its oppoſite ſide, as the tangent of this angle is to the tangent of the ſame ſide;* that is, $\dot{B} : \dot{\overline{AC}} :: \text{tang. } B : \text{tang. } AC$; or, $\dot{C} : \dot{\overline{AB}} :: \text{tang. } C : \text{tang. } AB$. Fig. 28.

DEMONSTRATION.

Since the angle at A and the oppoſite ſide, BC, of the triangle BAC remain conſtant, and the proportion between the *ſines* of angles and thoſe of their oppoſite ſides gives, $\text{ſin. } C = \frac{\text{ſin. } A}{\text{ſin. } BC} \times \text{ſin. } AB$, and, $\text{ſin. } B = \frac{\text{ſin. } A}{\text{ſin. } BC} \times \text{ſin. } AC$; it is evident that, the fluxions of ſin. C and ſin. B will be as thoſe of the *ſines* of the ſides, AB and AC, which are oppoſite to them. But, by *art.* 250, the fluxion of the arc which meaſures the angle C is to that of the arc AB as $\frac{\dot{\overline{\text{ſin. } C}}}{\text{coſ. } C}$ is to $\frac{\dot{\overline{\text{ſin. } AB}}}{\text{coſ. } AB}$, or, which is the ſame, $\dot{C} : \dot{\overline{AB}} :: \frac{\dot{\overline{\text{ſin. } C}}}{\text{coſ. } C} : \frac{\dot{\overline{\text{ſin. } AB}}}{\text{coſ. } AB}$; alſo, $\dot{\overline{\text{ſin. } C}} : \dot{\overline{\text{ſin. } AB}} :: \text{ſin. } C : \text{ſin. } AB$ (becauſe the relation of ſin. C : ſin. AB muſt be conſtant); wherefore, if ſin. C to *ſin.* AB be ſubſtituted for *ſin.* C : *ſin.* AB in the two laſt terms of the preceding proportion, and afterwards the *tangents* into which they may be changed, we ſhall get $\dot{C} : \dot{AB}$ :: *tang.* C : *tang.* AB. We might prove in the ſame manner that, $\dot{B} : \dot{AC}$ :: *tang.* B : *tang.* AC. Q. E. D.

COROLLARY I.

266. If A = 90°, we ſhall have, $\dot{C} : \dot{AB}$ :: R : *ſin.* AC, and alſo, $\dot{B} : \dot{AC}$ :: R : *ſin.* AB, *art.* 228.

COROLLARY II.

267. If the triangle be right-lined, the fluxions of the angles B and C will be equal, ſince the ſum of the three angles, as well as one of theſe angles, are conſtant quantities. Moreover, the *tangents* of the ſides, AB, AC, become then the ſides themſelves, and therefore we ſhall have, *As the fluxion of one of the angles is to that of its oppoſite ſide, ſo is the tangent of this angle to the ſame ſide*; that is, $\dot{C} : \dot{AB}$ :: *tang.* C : AB, or, ſince $\dot{C} = \dot{B}$, $\dot{B} : \dot{AB}$ :: *tang.* C : AB; and, $\dot{C} : \dot{AC}$ :: *tang.* B : AC.

THEOREM VI.

268. *Suppoſing ſtill one angle with its oppoſite ſide conſtant; the fluxions of the other two ſides will be as the coſines of their oppoſite angles, and thoſe of the other two angles as the coſines of their oppoſite ſides*; that is, $\dot{AB} : \dot{AC}$ :: *coſ.* C : *coſ.* B, and, $\dot{B} : \dot{C}$ :: *coſ.* AC : *coſ.* AB.

DEMONSTRATION.

Fig. 28. Having taken D*b* = DB, and D*c* = DC, it is manifeſt that the angles at *b* and *c* will be right, as alſo $b\beta = c\gamma$; and conſequently (by taking from theſe equal quantities the part *bc*, which is common),

mon), $\beta b = c\gamma$. This being premiſed, from the right-angled triangle βbB we ſhall get, Bβ : $b\beta$ :: R : *coſ*. B; and alſo from the right-angled triangle C$c\gamma$, $c\gamma$: Cγ :: *coſ*. C : R; and therefore, by the multiplication of correſponding terms, Bβ : Cγ :: *coſ*. C : *coſ*. B, or, $\overline{AB}$: $\overline{AC}$ :: *coſ*. C : *coſ*. B. Q. E. 1°. D.

2°. In the triangle DEF, whoſe parts are ſeverally the ſupplements to thoſe of the triangle BAC, we might find, DF : EF :: *coſ*. E : *coſ*. D; wherefore, by ſubſtituting for each term its correſponding value in the triangle BAC, we ſhall get, $\dot{B}$: $\dot{C}$ :: *coſ*. AC : *coſ*. AB. Q. E. 2°. D. Fig. 11.

COROLLARY I.

269. Since we have, $\overline{AB}$: $\overline{AC}$ :: *coſ*. C : *coſ*. B; we ſhall alſo have (by *art.* 229), *coſ*. AB × *ſin*. AB × R − *coſ*. AC × *coſ*. BC × *ſin*. AB : *coſ*. AC × *ſin*. AC × R − *coſ*. AB × *coſ*. BC × *ſin*. AC: and ſince we have, $\dot{C}$: $\dot{B}$ :: *coſ*. AB : *coſ*. AC, we ſhall likewiſe have, by putting for *coſ*. AC its value in *art*. 216, and dividing the two laſt terms by *coſ*. AB;

$$\dot{C} : \dot{B} :: R : \frac{cos.\ B \times sin.\ BC \times tang.\ AB}{R} + cos.\ BC.$$

COROLLARY II.

270. Since we have, by *Theo.* V. $\overline{AB}$: $\dot{C}$ :: *tang*. AB : *tang*. C; and, by the preſent Theorem, $\dot{C}$: $\dot{B}$ :: *coſ*. AB : *coſ*. AC; if we multiply the correſponding terms of theſe two proportions together, we ſhall get, $\overline{AB}$: $\dot{B}$ :: R × *ſin*. AB : *tang*. C × *coſ*. AC. By a ſimilar proceſs we might find, $\overline{AC}$: $\dot{C}$ :: R × *ſin*. AC : *tang*. B × *coſ*. AB. Then, if we ſubſtitute in theſe two proportions for *ſin*. AB and *ſin*. AC, their reſpective values, $\frac{sin.\ AC \times sin.\ C}{sin.\ B}$

and $\frac{\text{ſin. }AB \times \text{ſin. }B}{\text{ſin. }C}$, the following analogies will ariſe, $\dot{\overline{AB}} : \dot{B} :: \text{tang. }AC \times \text{coſ. }C : R \times \text{ſin. }B$, and, $\dot{\overline{AC}} : \dot{C} :: \text{tang. }AB \times \text{coſ. }B : R \times \text{ſin. }C$. Likewiſe, if we again ſubſtitute, in the firſt proportion, $\frac{\text{ſin. }AC \times R}{\text{tang. }AC}$ for coſ. AC, we ſhall find, $\dot{\overline{AB}} : \dot{B} :: \text{tang. }AC \times \text{ſin. }AB : \text{tang. }C \times \text{ſin. }AC :: \text{tang. }AC \times \text{coſ. }AB - \text{ſin. }AC \times \text{coſ. }BC : \frac{\text{ſin. }B \times \text{ſin. }AC \times \text{ſin. }BC}{R}$, by putting for coſ. C its value given in art. 229. —— Moreover, if we ſuppoſe the angle at A right, we ſhall get, from the latter proportion, $\dot{\overline{AC}} : \dot{C} :: \frac{1}{2}\text{ſin. }2AC : R \times \text{cot. }C$; by putting for tang. B and coſ. AB, their values in art. 228, $\frac{\text{cot. }C \times R}{\text{coſ. }BC}$ and $\frac{\text{coſ. }BC \times R}{\text{coſ. }AC}$, and for ſin. AC × coſ. AC the expreſſion which is equal to it, by art. 14.

COROLLARY III.

271. If the triangle be right-lined, the fluxions of the ſides will ſtill be as the *coſines* of their oppoſite angles; and the fluxions of the angles as the *coſines* of their oppoſite ſides. But, ſince in this caſe the ſides are, or may be, conſidered as arcs indefinitely ſmall, their *coſines* will be all equal, and therefore the fluxions of the angles alſo equal; a thing that we may be eaſily convinced of by other different conſiderations.

SECT. IV.

Concerning the fluxions of a ſpherical or right-lined triangle, when two of its ſides remain conſtant.

THEOREM VIII.

272. *The fluxion of the angle contained between the two conſtant ſides is to the fluxion of either of the other*

two angles, as the product of the ſine total and ſine of the variable ſide is to the product of the ſine of the ſide oppoſite to the latter angle and the coſine of the third angle adjacent to this ſide; that is, by ſuppoſing the ſides AB and AC conſtant, $\dot{A} : \dot{B} :: R \times \textit{ſin.}\ BC : \textit{ſin.}\ AC \times \textit{coſ.}\ C$, and, $\dot{A} : \dot{C} :: R \times \textit{ſin.}\ BC : \textit{ſin.}\ AB \times \textit{coſ.}\ B$. *Fig.* 29.

DEMONSTRATION.

Suppoſing the angle BAC changed to BAc, let the ſides, AC, Ac, be produced till they become quadrants, ACF, Acf; and it is manifeſt that Ff will be the meaſure of the variation of the angle at A; in like manner, let the ſides, BC, Bc, be produced to, G, g, ſo that, BCG, Bcg, may be each 90°; and it is again manifeſt that, Gg will be the meaſure of the variation of the angle at B. Moreover, let there be taken upon the ſide Bc the arc Bγ = BC, and there will by this means be formed the triangle Cγc, right angled at γ, wherein the angle γcC is perceptibly equal to the complement of the angle C. This being premiſed, on account of the ſimilar ſectors, Ff, Cc we ſhall have, $Ff : Cc :: R : \textit{ſin.}\ AC$; alſo, on account of the right-angled triangle Cγc, $Cc : C\gamma :: R : \textit{coſ.}\ C$; and, on account of the ſimilar ſectors, Gg, Cγ, $C\gamma : Gg :: \textit{ſin.}\ BC : R$; then, if we multiply the correſponding terms of theſe three proportions together, and ſubſtitute in the reſulting product the values of Ff and Gg, we ſhall get, $\dot{A} : \dot{B} :: R \times \textit{ſin.}\ BC : \textit{ſin.}\ AC \times \textit{coſ.}\ C$. —— We might prove by a ſimilar proceſs that, $\dot{A} : \dot{C} :: R \times \textit{ſin.}\ BC : \textit{ſin.}\ AB \times \textit{coſ.}\ B$. Q.E.D.

COROLLARY I.

273. If in the firſt proportion we put $\frac{\textit{ſin.}\ BC \times \textit{ſin.}\ B}{\textit{ſin.}\ A}$ for *ſin.* AC, we ſhall find, $\dot{A} : \dot{B} :: R \times \textit{ſin.}\ A :$

ſin. B × coſ. C :: ſin. BC × tang. C : ſin. AC × ſin. C (by multiplying the two laſt terms by $\frac{\text{ſin. C}}{\text{coſ. C}}$) :: ſin. A × tang. C : ſin. B × ſin. C (by ſubſtituting for ſin. BC its value $\frac{\text{ſin. AC} \times \text{ſin. A}}{\text{ſin. B}}$) :: ſin. BC × tang. C : ſin. B × ſin. AB (by taking away ſin. C in one of the preceding compariſons) :: ſin.² BC : coſ. AB × R — coſ. AC × coſ. BC (by putting in the firſt proportion the value of coſ. C, given in *art.* 225) :: ſin. AB × ſin. A : ſin. AC × ½ ſin. 2C (by ſubſtituting in the ſame proportion $\frac{\text{ſin. AB} \times \text{ſin. A}}{\text{ſin. C}}$ for ſin. BC, and ½ ſin. 2C for $\frac{\text{coſ. C} \times \text{ſin. C}}{\text{R}}$) :: R³ : ſin.² B × coſ. AB — ſin. B × coſ. B × cot. A (by ſubſtituting $\frac{\text{ſin. BC} \times \text{ſin. B}}{\text{ſin. A}}$ for ſin. AC; coſ. AB × ſin. A × ſin. B — coſ. A × coſ. B × R (*art.* 221) for coſ. C, and cot. A for $\frac{\text{R} \times \text{coſ. A}}{\text{ſin. A}}$) :: R³ : coſ. AB × RR — ſin. AB × coſ. B × cot. BC; by taking away cot. A (*art.* 214), and putting R² for ſin.² B + coſ.² B.

COROLLARY II.

274. Similar ſubſtitutions in the ſecond proportion will give the following compariſons; A : C :: R × ſin. A : ſin. C × coſ. B :: ſin. BC × tang. B : ſin. AB × ſin. B :: tang. B × ſin. BC : ſin. AC × ſin. C :: tang. B × ſin. A : ſin. B × ſin. C :: ſin.² BC : coſ. AC × R — coſ. AB × coſ. BC.

COROLLARY III.

275. If the triangle be right-lined, the *ſines* of the ſides will then become the ſides themſelves, and therefore we ſhall have, A : B :: R × BC : AC × coſ. C, and A : C :: R × BC : AB × coſ. B; that is to ſay, *The fluxions of the angle included between* [illegible] *will*

will be in the direct compound ratio of the ſides oppoſite to theſe angles and of the radius to the coſine of the third angle.

THEOREM VIII.

276. *The fluxion of the angle contained between the conſtant ſides is to that of its oppoſite ſide, as radius ſquared is to the rectangle under the ſine of either of the other angles and that of its adjacent ſide;* that is, $\dot{A} : \dot{\overline{BC}} :: R^2 : \text{ſin. } C \times \text{ſin. } AC :: R^2 : \text{ſin. } B \times \text{ſin. } AB$. Fig. [illegible]

DEMONSTRATION.

The ſimilar ſectors, Ff, Cc, give (as we have already obſerved), $Ff : Cc :: R : \text{ſin. } AC$, and the little right-angled triangle Crc, $Cc : rc :: R : \text{ſin. } C$; wherefore, by multiplying correſponding terms together, we ſhall get, $Ff : rc :: R^2 : \text{ſin. } AC \times \text{ſin. } C$; or, $\dot{A} : \dot{\overline{BC}} :: R^2 : \text{ſin. } AC \times \text{ſin. } C$. —— By ſuppoſing the angle A to flow towards B, we ſhould find in like manner, $\dot{A} : \dot{\overline{BC}} :: R^2 : \text{ſin. } AB \times \text{ſin. } B$. Q. E. D.

COROLLARY I.

277. If we ſubſtitute $\frac{\text{ſin. } BC \times \text{ſin. } B}{\text{ſin. } A}$ and $\frac{\text{ſin. } AB \times \text{ſin. } B}{\text{ſin. } C}$ for ſin. AC, we ſhall get, $\dot{A} : \dot{\overline{BC}} :: R^2 \times \text{ſin. } A : \text{ſin. } BC \times \text{ſin. } B \times \text{ſin. } C :: R^2 \times \text{ſin. } C : \text{ſin. } AB \times \text{ſin. } B \times \text{ſin. } C :: R^2 \times \text{ſin. } BC \times \text{ſin. } C : \text{ſin.}^2 AB \times \text{ſin. } A \times \text{ſin. } B$, by putting $\frac{\text{ſin. } AB \times \text{ſin. } A}{\text{ſin. } BC}$ for ſin. C.

COROLLARY II.

278. If the triangle be right-lined, we ſhall have, $\dot{A} : \dot{\overline{BC}} :: R^2 : AC \times \text{ſin. } C :: R^2 : AB \times \text{ſin. } B :: R : AD$ (by ſuppoſing AD to be a perpendicular let fall from the angle A upon the oppoſite ſide, BC); whence we infer that, if two ſides of a right-lined triangle be conſtant, *The fluxion of the*

the angle included between theſe ſides will be to that of its oppoſite ſide, as radius is to the perpendicular let fall from this angle upon the ſaid ſide.

THEOREM IX.

279. *The function of either of the angles adjacent to the variable ſide is to the function of this ſide, as the co-tangent of the other adjacent angle is to the ſine of the*

Fig. 29. *ſaid ſide*; that is, C : $\overline{BC}$ :: *cot.* B : *ſin.* BC, and B : $\overline{BC}$ :: *cot.* C : *ſin.* BC.

DEMONSTRATION.

Since we have, by *art.* 272, A : C :: R × *ſin* BC : *ſin.* AB × *coſ.* B, and, by *art.* 276, $\overline{BC}$: A :: *ſin.* C × *ſin.* AC : R²; if we multiply theſe two proportions together, we ſhall get $\overline{BC}$ × A : C × A :: *ſin.* BC × *ſin.* C × *ſin.* AC : R × *ſin.* AB × *coſ.* B :: *ſin.* BC : $\frac{\text{ſin. AB} \times \text{coſ. B} \times \text{R}}{\text{ſin. C} \times \text{ſin. AC}}$ :: *ſin.* BC : *cot.* B, by ſubſtituting $\frac{\text{ſin. AC} \times \text{ſin. C}}{\text{ſin. B}}$ for *ſin.* AB, and *cot.* B for $\frac{\text{coſ. B} \times \text{R}}{\text{ſin. B}}$; and conſequently, $\overline{BC}$: C :: *ſin.* BC : *cot.* B, or, *invertendo*, C : $\overline{BC}$ :: *cot.* B : *ſin.* BC. Q. E. D.

COROLLARY I.

280. Since, B : $\overline{BC}$ :: *cot.* C : *ſin.* BC, it will be eaſy to obtain the following compariſons; :: R × *coſ.* C : *ſin.* AB × *ſin.* A (by putting $\frac{\text{ſin. A} \times \text{ſin. AB}}{\text{ſin. C}}$ for *ſin.* BC) :: R × R : *tang.* C × *ſin.* BC (by putting $\frac{\text{R}^2}{\text{tang. C}}$ for *cot.* C) :: *cot.* C × *ſin.* B : *ſin.* AC × *ſin.* A (by ſubſtituting $\frac{\text{ſin. AC} \times \text{ſin. A}}{\text{ſin. B}}$ for *ſin.* BC) :: R : $\frac{\text{ſin. BC} \times \text{ſin. B} \times \text{R}}{\text{cot. AB} \times \text{ſin. BC} - \text{coſ. B} \times \text{coſ. BC}}$ (by *art.* 214) :: *cot.* AB × *ſin.* BC − *coſ.* B × *coſ.* BC : *ſin.* BC × *ſin.*

ſin. B : $\frac{\text{cot.}AB\times R}{\text{ſin.}B} - \frac{\text{cot.}BC\times R}{\text{tang.}B}$: R (by dividing by ſin. BC × ſin. B, &c.) :: cot. AB $- \frac{\text{coſ.}B\times\text{cot.}BC}{R}$: ſin. B, by multiplying by ſin. B, &c. — We might find comparisons exactly similar for the proportion, $\dot{C}$: $\dot{BC}$ :: cot. B : ſin. BC; but think it unneceſſary to particularize them.

COROLLARY. II.

281. Since, $\dot{B}$: $\dot{BC}$ :: cot. C : ſin. BC, and $\dot{BC}$: $\dot{C}$ :: ſin. BC : cot. B, if we multiply the correſponding terms of theſe two proportions together, we ſhall find, $\dot{B}$: $\dot{C}$ :: tang. B : tang. C :: $\frac{\text{ſin.}B}{\text{coſ.}B}$: $\frac{\text{ſin.}C}{\text{coſ.}C}$ :: ſin. B × coſ. C : ſin. C × coſ. B :: ſin. AC × coſ. C : ſin. AB × coſ. B, by ſubſtituting $\frac{\text{ſin.}AB\times\text{ſin.}B}{\text{ſin.}AC}$ for ſin. C.

COROLLARY. III.

282. If the triangle be right-lined, the Theorem will give, $\dot{BC}$: $\dot{C}$ :: BC : cot. B :: BC × ſin. B : coſ. B × R; or, as the perpendicular let fall from the angle C upon the ſide AB is to the coſine of the angle B. In like manner, $\dot{BC}$ will be : $\dot{B}$ :: BC : cot. C, and laſtly, $\dot{B}$: $\dot{C}$ :: tang. B : tang. C.

SECT. V.

Concerning the fluxions of any ſpherical Triangle, whereof two of the angles are ſuppoſed conſtant.

THEOREM X.

283. *Suppoſing the two angles, B, C, in the ſpherical triangle* BAC *conſtant; the fluxion of the ſide to which theſe angles are adjacent will be to that of either* Fig. 28.

of the other sides, in the direct compound ratio of the sines of the angles opposite to these sides and of the radius to the cosine of the third side; that is, $\dot{\overline{BC}} : \dot{\overline{AB}}$ :: *ſin.* A × R : *ſin.* C × *coſ.* AC.

DEMONSTRATION.

Fig. 11. In the triangle DEF, which has all its parts ſupplemental to thoſe of the triangle BAC, the two ſides, DF, FE, will be variable: whence this caſe appears to be reduced to that of the firſt Theorem of the laſt Section; therefore, we ſhall have, $\dot{F} : \dot{D}$ :: R × *ſin.* DE : *ſin.* FE × *coſ.* E; and, by taking the correſponding values in the triangle BAC, $\dot{\overline{BC}} : \dot{\overline{AB}}$ :: R × *ſin.* A : *ſin.* C × *coſ.* AC. We might prove in like manner that, $\dot{\overline{BC}} : \dot{\overline{AC}}$ :: R × *ſin.* A : *ſin.* B × *coſ.* AB. Q.E.D.

THEOREM XI.

284. *The fluxion of the ſide contained between the conſtant angles is to that of the oppoſite angle, as the coſecant of either of the other ſides is to the ſine of the conſtant angle adjacent to this ſide*; that is, $\dot{\overline{BC}} : \dot{A}$ :: *coſec.* AB : *ſin.* B :: *coſec.* AC : *ſin.* C.

DEMONSTRATION.

It is well known, that the *coſecant* of an arc is equal to the ſquare of the radius divided by the *ſine* of this arc; therefore, the whole is reduced to prove that, $\dot{\overline{BC}} : \dot{A}$ :: R^2 : *ſin.* B × *ſin.* AB; a thing eaſily done by means of the triangle DEF and *Theo.* VIII. by purſuing a method ſimilar to that uſed in the demonſtration of the laſt Theorem. Q.E.D.

THEOREM XII.

285. *The fluxion of either of the ſides oppoſite to the conſtant angles is to the fluxion of the third angle, as the cotangent of the other ſide is to the ſine of the ſaid*

angle; that is, AB : A :: *cot.* AC : *ſin.* A, and AC : A :: *cot.* AB : *ſin.* A.

DEMONSTRATION.

This propoſition may be immediately deduced from *Theo.* IX. by applying it to the triangle DEF, and making the neceſſary changes. Q. E. D.

COROLLARY I.

286. It follows from the laſt Theorem, that the fluxions of the ſides, AB, AC, oppoſite to the conſtant angles, are as the *tangents* of theſe ſides.

COROLLARY II.

287. By ſuppoſing the triangle right-lined, the firſt Theorem ſhews us, that the fluxions of the ſides are to each other as the *ſines* of the angles oppoſite to theſe ſides; which thing is alſo well known by, and eaſily proved from, the elements of Geometry. From the ſecond Theorem nothing can be deduced, ſince in a plain triangle, which is ſuppoſed to have two of its angles conſtant, A cannot obtain. The ſame aſſertion may be made with reſpect to the third Theorem. Laſtly, from the preceding Corollary it may be inferred, that the fluxions of the ſides are as the ſides themſelves; a thing likewiſe demonſtrable by the elements of plain Geometry.

OBSERVATION.

288. The foregoing Theorems might have been eaſily demonſtrated from the figures uſed in the projections, by finding the evaneſcent relations of the different lines therein deſcribed. They might likewiſe have been deduced from the formulæ which have been obtained by the application of Algebra, by ſuppoſing ſome of the letters, *a*, *b*, *c*, *d*, *f*, *g*, &c. variable, according to the conditions

of the Theorem to be demonſtrated, and then paſſing from the fluxions of the *ſines*, *tangents*, &c. to thoſe of the arcs of a circle: but the method of Mr. *Cotes* appeared to me the cleareſt, and at the ſame time the eaſieſt; for which reaſon I have not given much more than a tranſlation of his excellent little Treatiſe, "*De æſtimatione errorum in mixta Matheſi:*" the theory indeed I have attempted to render ſomewhat more ſimple, by applying it to plain triangles in bare Corollaries; but ſuch a liberty I hope my readers will candidly excuſe.—— What now remains is to make ſome applications of this theory to different examples.

EXAMPLE I.

Fig. 30. 289. *Let there be an object whoſe height,* AB, *we would meaſure by taking with an inſtrument the angle* ACB; *it is required to determine what error may be committed with reſpect to the ſaid height, by ſuppoſing the error in the meaſure of the obſerved angle to be known.*

SOLUTION.

It is manifeſt that, in the triangle BAC, the ſide AC and right angle, A, remain conſtant; therefore, by *art.* 258, we ſhall have, AB : C :: BC : *ſin.* ABC :: 2AB : *ſin.* 2ACB (for by ſubſtituting for *ſin.* 2ACB its value $\frac{2\,\textit{ſin.}\ \mathrm{ACB} \times \textit{ſin.}\ \mathrm{ABC}}{\mathrm{R}}$ by *art.* 13, this proportion becomes, BC : R :: AB : *ſin.* ACB); and conſequently, *The error of the height* AB *is to the meaſure of the error of the angle* C, *as the double of this height is to the ſine of the double of the obſerved angle.* The error therefore, which can prevail in the determination of the ſaid height, will be the leaſt poſſible, when the *ſine* of double the obſerved angle is the greateſt poſſible; that is, when this angle is 45°: for which reaſon, when we would find the height, AB, of any object, by the obſervation of any angle, as C, we muſt ſo contrive it, that the angle obſerved may be the neareſt 45° poſſible. Q. E. I.

290. Let us now proceed to diſcover what the error of the height AB may be, upon a ſuppoſition that a miſtake of one minute is committed in the determination of the angle C. If we make the radius = 10000000, the arc of a minute, which meaſures the ſuppoſed error, will be 2909; the double whereof is 5818: but this quantity is to the *ſine* of double the angle of 45° or to radius, as 1 to 1719, and therefore, according to this ſuppoſition, the miſtake concerning the ſaid height AB will be $\frac{1}{1719}$ part of the height itſelf. —— If the error of the angle, ACB, be increaſed or diminiſhed, the error of the height will be increaſed or diminiſhed in the ſame ratio: alſo, if the angle obſerved be greater or leſs than 45°, the error of the height will be increaſed in the ratio of radius to the *ſine* of double the ſaid angle.

EXAMPLE II.

291. *It is required to find the hour of the day or night, by obſerving the altitude of a ſtar; and alſo to aſſign the error of time, by ſuppoſing the error in the obſerved altitude to be known.*

SOLUTION.

Let there be a ſpherical triangle PZS, wherein P is the pole, Z the zenith of the place of obſervation, and S the place of the ſtar obſerved; then will PS be the complement of the ſtar's declination; ZS the complement of its altitude; PZ the complement of the latitude, and the angle ZPS a variable hour-angle contained betwixt the conſtant ſides, PZ, PS. This being premiſed, the fluxion of the hour-angle SPZ is (by art. 276) to the fluxion of the oppoſite ſide SZ, as the *coſecant* of the *Fig.* 31.

the angle PZS is to the *ſine* of the ſide PZ; or, $\dot{P} : \dot{\overline{ZS}} :: R^2 : \text{ſin.}\, Z \times \text{ſin.}\, PZ$: but the variation of the angle at P is the meaſure of the error in time; therefore, by taking $\dot{P}$ for the ſaid error, we ſhall get, $\dot{P}$ or $\dot{t} = \frac{R^2 \times \dot{\overline{ZS}}}{\text{ſin.} Z \times \text{ſin.} PZ}$. Q. E. I.

COROLLARY.

292. It follows from hence that, if we ſuppoſe the ſame error to prevail in different obſervations, and the latitude of the place to continue alſo the ſame, the error of time will not be altered, whatever the altitude of the ſtar in the given vertical be; ſince $\frac{R^2}{\text{ſin.} Z}$ will be always a conſtant quantity. Moreover, we may perceive that, by ſtill ſuppoſing the latitude conſtant, the error of time will be the leaſt, when the ſtar is obſerved on the prime vertical; and, if we aſſume two different latitudes, the error will be the leaſt poſſible, when the obſervation is made at the equator, and the ſtar ſituate on the prime vertical. For inſtance (granting the truth of theſe two aſſertions); if we ſuppoſe a miſtake of one minute to be committed in the obſervation of the altitude, we ſhall find, by making the calculation, that the error in time is 4″: but, if we again ſuppoſe the obſerver to be removed to a certain diſtance from the equator, the error then (provided the ſtar be ſtill obſerved on the prime vertical) will be to that aforegoing, in the ratio of radius to the *coſine* of the latitude; ſo that for a latitude of 45°, the error will be $5''\frac{2}{3}$; for a latitude of 50°, $6''\frac{2}{9}$, and for a latitude of 55°, $6''\frac{37}{38}$. Laſtly, if a ſtar be obſerved on any vertical inclined to the meridian, the error will be yet increaſed with regard to thoſe above given, in the ratio of radius to the *ſine* of the angle contained between them;

and its variations at the same time depend on those supposed in the first observation.

EXAMPLE III.

293. *Let*, LSR, L'S'R', *be two almacanters or circles parallel to the horizon, and* S, S', *the points where a ſtar (whoſe declination is ſuppoſed the ſame during the whole day) paſſes over theſe circles; it is required to find what the relation of the angles*, PSZ, PS'Z, *made by the hour-circles*, PS, PS', *with their correſponding verticals, ought to be, ſo that the time taken up by the ſtar in paſſing from one almacanter to the other may be the leaſt poſſible.* Fig. 32.

SOLUTION.

From the *data* of the Problem it is manifeſt that, in the triangles, PZS, PZS', the ſides, PZ, ZS and ZS', remain conſtant, whilſt the ſides, PS, PS', vary. Let then h and h' be put for the hour-angles at P; S and S' for the angles at the ſtar at S and S', made by the hour-circles PS and PS' with the correſponding verticals ZS and ZS', and let D repreſent the declination; the fluxion whereof, $\dot{D}$, will be negative, ſince, if the declination be northerly (as we have at preſent ſuppoſed it), the arcs, PS, PS', will decreaſe as the declination increaſes. This being premiſed, we ſhall have, by *art.* 279, $\dot{h} = \frac{cot.\, S \times \dot{D}}{cos.\, D}$, and, in like manner, $\dot{h}' = \frac{cot.\, S' \times \dot{D}}{cos.\, D}$; then, ſince the time over the arc SS' is to be a *minimum*, the difference of theſe fluxions of the hour-angles muſt be equal to nothing; or, $\frac{cot.\, S \times \dot{D}}{cos.\, D} - \frac{cot.\, S' \times \dot{D}}{cos.\, D} = 0$: whence we immediately deduce, *cot.* S = *cot.* S'; and there-

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* For theſe two Examples ſee *Cotes*'s aforementioned Treatiſe.

fore, $S = S'$. Now, if this ſtar be the ſun, and one of the almacanters coincide with the horizon, whilſt the other becomes the crepuſcular circle (which is uſually ſuppoſed to be 18° below the horizon), the conſequence juſt inferred intimates to us, that, on a day when the thing required happens, the angles at the ſun, both at the horizon and crepuſcular circle, are equal. From this conſequence therefore, the day of ſhorteſt twilight may be found, provided we firſt ſeek what the declination anſwering to this determinate relation of the angles at the ſun ought to be: which we proceed to do in the following

PROBLEM.

294. *Suppoſing the angles, made by two hour-circles with their correſponding verticals, equal; it is required to find the ſun's declination anſwering thereto.*

SOLUTION.

Fig. 32. Let S and S′ repreſent the angles contained between the hour-circles and their correſponding verticals; C and C′ the diſtances of two almacanters from the horizon; L the latitude of the place, and D the ſun's declination: then (by *art.* 225) we ſhall get from the triangle PZS, $\frac{\textit{coſ.}\,D \times \textit{coſ.}\,S}{R} = \frac{R \times \textit{ſin.}\,L - \textit{ſin.}\,C \times \textit{ſin.}\,D}{\textit{coſ.}\,C}$, and $\frac{\textit{coſ.}\,D \times \textit{coſ.}\,S'}{R} = \frac{R \times \textit{ſin.}\,L + \textit{ſin.}\,C' \times \textit{ſin.}\,D}{\textit{coſ.}\,C'}$ from the triangle PZS′; putting + before *ſin.* C′ in the ſecond formula, becauſe ZS′, as well as its oppoſite ſide, is obtuſe. Now the firſt members of theſe equations being equal, the ſecond will be ſo likewiſe, and give this new equation, $R \times \textit{ſin.}\,L \times \textit{coſ.}\,C' - \textit{ſin.}\,C \times \textit{ſin.}\,D \times \textit{coſ.}\,C' = R \times \textit{ſin.}\,L \times \textit{coſ.}\,C + \textit{ſin.}\,C' \times \textit{ſin.}\,D \times \textit{coſ.}\,C$: whence we immediately deduce, $\textit{ſin.}\,C' \times \textit{coſ.}\,C + \textit{ſin.}\,C \times \textit{coſ.}\,C' : R \times \overline{\textit{coſ.}\,C' - \textit{coſ.}\,C} ::$ *ſin.*

ſin. L : ſin. D; or, by *art.* 85, ſin. $\overline{C'+C}$: coſ. C′ — coſ. C :: ſin. L : ſin. D, or otherwiſe, by *art.* 87 and 94, coſ. $\frac{C+C'}{2}$: ſin. $\frac{C-C'}{2}$ :: ſin. L : ſin. D. Then, if we ſuppoſe C = 0 (as it really muſt be in the caſe of the ſhorteſt twilight), we ſhall get this laſt proportion; R : — *tang.* $\frac{C'}{2}$ :: ſin. L : ſin. D; that is to ſay, *As radius is to the tangent of half the depreſſion of the crepuſcular circle below the horizon, ſo is the ſine of the latitude to the ſine of the ſun's declination:* where obſerve, that, as the declination is negative (becauſe *tang.* $\frac{C'}{2}$ is preceded by —), it muſt be of a denomination contrary to the latitude of the place.

OLD METHOD *of* SOLUTION.

295. Let u be put for the *coſine* of the hour-angle at the inſtant when the twilight ends, and the angle itſelf will be repreſented by, $\int \times \frac{r\dot{u}}{\sqrt{r^2-u^2}}$; alſo, let z be put for the *coſine* of the ſemi-diurnal arc, and this arc will be, $\int \times \frac{r\dot{z}}{\sqrt{r^2-z^2}}$: then muſt $\frac{\dot{u}}{\sqrt{r^2-u^2}} - \frac{\dot{z}}{\sqrt{r^2-z^2}} = 0$. Moreover, let c be put for the *ſine* of the latitude, and g for its *coſine*; b for the *ſine* of the depreſſion of the crepuſcular circle below the horizon, and y for the *ſine* of the ſun's declination; and we ſhall have, $u = \frac{br^2 - cry}{g\sqrt{r^2-y^2}}$ and, $z = \frac{cry}{g\sqrt{r^2-y^2}}$; then, if by means of theſe values we exterminate u and z in the firſt equation, and afterwards divide by y, we ſhall get, $by^4 + 2cry^3 - g^2by^2 - 2cr^3y - c^2br^2 = 0$; whence, $y = r$;

If ZS′ & its opposite ∠ be acute the proportion will be cos. $\frac{C-C'}{2}$: sin. $\frac{C+C'}{2}$ :: sin. L : sin. D.

But of theſe (four) values of x, the two firſt, indicating the ſun to be at the pole (where it can never phyſically arrive), are foreign to the queſtion and unapplicable: of the other two, the greateſt gives the *minimum* of the ſum of the ſemi-diurnal arc,* and the arc which the ſun paſſes over from noon to the end of twilight; whilſt the leaſt gives the *minimum* of the difference of theſe arcs: conſequently, the declination required hath for its ſine, $x = \frac{-cr + c\sqrt{r^2 - b^2}}{b}$. Therefore, if we put t for the *tangent* of half the depreſſion of the crepuſcular circle below the horizon, we ſhall have, by *art.* 75, $x = -\frac{ct}{r}$, or, $r : t :: c : -x$; which ſhews, as before, that the ſun's declination is of a denomination contrary to the latitude of the place*.

OBSERVATION.

296. From the firſt of the two preceding ſolutions it follows that, if any two ſpherical triangles have two ſides equal each to each, and alſo an equal angle, oppoſite to one of the equal ſides, we ſhall have this analogy; *As the coſine of the ſide adjacent to the equal angle is to the coſine of the ſide oppoſite to the ſame angle, ſo is the coſine of half the ſum of the other two ſides to the coſine of half their difference.* —— This proportion is immediately deduced from, coſ. $\frac{C+C'}{2}$: ſin. $\frac{C-C'}{2}$ :: ſin. L : ſin. D; by obſerving that the particular denominations in the Problem will be rendered general, if ſuch things only be preſerved as are eſſential to the triangles we have occaſion to conſider.

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* Should not this ſolution appear entirely perſpicuous, conſult *Maupertuis*'s Aſtronomie Nautique, p. 38, &c.

If, in like manner, it was required to find what the declination of a ſtar ought to be, ſo that, in paſſing over the interval contained between two given hour-circles, its change in altitude might be the *greateſt* poſſible; we ſhould get by a ſimilar proceſs, and with the preceding denominations, $Cof. \frac{b'-b}{2} : cof. \frac{b+b'}{2} :: tang. L : tang. D.$*

EXAMPLE IV.

297. *To find at all times the correction neceſſary to be made for noon-day, as deduced from correſponding altitudes of the ſun, whoſe declination may be ſuppoſed to undergo a little change during the interval of two equal altitudes.*

SOLUTION.

Aſtronomers (in order to be certified of the inſtant of noon) take the altitude of the ſun's centre ſome time in the morning, then obſerve at what time in the afternoon it comes to the ſame altitude, and afterwards take half the interval of time elapſed between theſe two obſervations, as marked by a clock, for the inſtant of the ſun's paſſage over the meridian. Now this method would be quite exact and rigorous, provided the ſun did not change its declination during the interval of the obſervations; but this is a thing never ſtrictly true. At the time of the ſolſtices this error is the leaſt poſſible, becauſe then the ſun does not ſenſibly change its declination for two or three days; but at all other times of the year the operation abovementioned wants a correction, which it is required by this Problem to determine: for, it is evident that, the ſun when in the aſcending ſigns arrives later at the ſame altitude, and ſooner when in the deſcending ſigns; and therefore, in the ſix firſt months ſomething muſt be ſubtracted, and in the ſix other months ſomething added, that the preciſe inſtant of noon-day may be obtained. —— Now to find this correction; let P*s* be an hour-circle very near to PS, and terminated by the ſame almacanter LSK: then, it is manifeſt that, the angle SP*s*, reduced into time, will expreſs how much later, or ſooner, the ſun arrives at the ſame altitude in the afternoon than in the morning; and conſequently, the half thereof will be the time which muſt be ſubtracted from, or added to, the inſtant of noon deduced from correſponding altitudes. This being premiſed, if the change in declination be ſuppoſed ſo ſmall, as to coincide with the naſcent variation or fluxion of the arc PS; it is evident that, the ſides, PZ, ZS, of the triangle PZS will remain conſtant, whilſt the ſide PS varies. But in *art.* 280 we inferred that, $\dot{B} : \dot{\overline{BC}} :: \frac{\cot.\ AB}{\sin.\ B} \pm \frac{\cot.\ BC}{\tan g.\ B} : R$; and therefore, by ſuppoſing the points, B, A, C, to repreſent, P, Z, S, we ſhall have, $\dot{P} = \frac{\dot{\overline{PS}}}{R} \times \overline{\frac{\tan g.\ lat.}{\sin.\ P} \pm \frac{\tan g.\ dec.}{\tan g.\ P}}$; a formula, exactly agreeing with that which Mr. *Maupertuis* hath given in p. 34 of his *Aſtronomie Nautique*: where obſerve, that the ſign + prevails, when the declination is ſoutherly, but the ſign —, when it is northerly; as we may be eaſily convinced from the ſolutions given in the preceding Chapter. —— The half of this expreſſion, reduced into time, will be the correction required *. Q. E. I.

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*] To me it appears that, as ZS′ and the angle ZPS′ are obtuſe, the proportion ought to be, $Sin. \frac{b+b'}{2} : Sin. \frac{b'-b}{2} :: tang. L : tang. D.$

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* See *Robertſon*'s Tranſlation of *De la Caille*'s Aſtronomy, p. 150.

They, who are deſirous of ſeeing a variety of other applications of fluxional analogies, may have recourſe to the *Memoirs of the Academy* for the year 1744, or to the *Aſtronomie* of Mr. *De la Lande.* It may be proper, however, to obſerve, that ſeveral of the ſolutions obtained by this means are not attended with a truly Geometrical ſtrictneſs; but approach ſo much the nearer to preciſion, as the arcs under conſideration approach the nearer to naſcent arcs. —— We ſhall now conclude this Chapter with the ſolution of a Problem which is more curious than uſeful (but may nevertheleſs intimate to us ſome important truths, by the ſimplicity of the ſolution whereat we arrive), viz. with finding the area of any ſpherical triangle. Several Geometers, amongſt whom may be reckoned Mr. *J. Bernoulli*, have inveſtigated the quadrature of this portion of the ſurface of the ſphere; taking for the element (indefinitely ſmall part) thereof a differential (fluxion), whoſe integral (fluent) depends on the quadrature of the hyperbola. The ſubjoined ſolution, which we have in a great meaſure deduced from the Works of Dr. *Wallis*, is indiſputably one of the moſt elegant that can be given, and may furniſh us with methods of obtaining, by the quadrature of the circle, the fluents of ſeveral fluxions which appear very complicated.

PROBLEM.

298. *To find the area of any ſpherical triangle,* BAC. *Fig.* 33.

SOLUTION.

In the firſt place, let the ſides, AB, BC, containing any angle, B, be produced, ſo that, BA*b*, BC*b*, AB*a*, CB*c*, may be arcs of 180°; and, it is manifeſt that, the triangles, A*b*C, *a*B*c*, will be equal in all reſpects. In the next place, let a plane be conceived to ~~paſs through the points, A, C; *a*, *c*,~~

and of conſequence through the centre of the ſphere; and the ſurface ACB*ac* will be equal to that of an hemiſphere; which we ſhall denote by s, putting the radius of its great circle $= r$, and circumference $= c$. This done, if the triangles, ABC, CB*a*, *a*B*c*, *c*BA and A*b*C, be repreſented by, M, O, N, P and *n*, reſpectively, the portion of the ſpherical ſurface *b*ABC*b* will be determined by this analogy, as $c : B :: 2s : \frac{2sB}{c} = M + n$ or $M + N$; the portion AC*c*BA by, $c : A :: 2s : \frac{2sA}{c} = M + O$, and that of CB*c*AC by, $c : C :: 2s : \frac{2sC}{c} = M + P$: then, if we collect theſe three equalities, we ſhall have, $\overline{B + A + C} \times \frac{2s}{c} = 3M + N + O + P$; whence, as $M + N + O + P = s$, M, or the area of the triangle BAC, will be found $= \overline{B + A + C} \times \frac{s}{c} - \frac{1}{2}s$; or, as the ſurface of an hemiſphere $= cr$, it will be otherwiſe found $= \overline{A + B + C} \times r - \frac{1}{2}cr = \overline{A + B + C - \frac{1}{2}c} \times r$; and therefore, if from the ſum of the three angles of any ſpherical triangle 180° be ſubtracted, and the remainder multiplied by the radius ($= 57°.2957795$); the product will give the ſurface of ſuch triangle in ſquare degrees*.

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* See *Emerſon's* Trigonometry, p. 197, 2d edition.

CHAP. VI.

Application of the formulæ demonſtrated in the ſecond Chapter, to ſeveral Problems of Aſtronomy.

THough, by the Tables annexed to the ſecond Chapter, our readers might be enabled to ſolve all the Problems, which can be reduced to any of the caſes of right or oblique-angled ſpherical triangles; yet, as they, who are not well verſed in the performance of trigonometrical calculations, might be afraid of committing miſtakes upon applying the rules therein given to the Logarithms; we thought ourſelves under an indiſpenſible obligation of annexing a proper number of examples, in order to render this little Treatiſe as complete as poſſible. We might, like all other Writers upon Trigonometry, have made the applications to triangles, whoſe parts ſhould have had no particular denomination; but we judged it better to conſider the ſeveral triangles as relating to the circles of the ſphere, that the ſolutions thence derived might become more intereſting to Learners.——We will ſuppoſe then throughout this Chapter the principal circles of the ſphere to be denoted, as they are exhibited in fig. 34. viz. HOR to be the horizon; AOQ the equator; IEL the ecliptic, making with the equator an angle that diminiſhes very ſlowly, and which, with moſt other Aſtronomers, we will here imagine to be 23° 28′ 30″; and PZH the meridian, wherein, P, *p*, repreſent the poles, viz. P the pole which is elevated above the horizon, and *p* that depreſſed below. The altitude of the pole we will ſuppoſe to be 40° 51′, as it is for the City of *Paris*; and, in the laſt place, denote any ſtar or its place by S.——We take it for granted that the reader is perfectly acquainted with the meaning of a ſtar's declination, right aſcenſion, longitude, latitude, &c. as well as with all the principal circles of the ſphere; abundant explanations whereof may be met with in a variety of Treatiſes on that ſubject.

PROBLEM I.

Fig. 34. 299. *Given the ſun's place in the ecliptic; to find his declination, or diſtance, SD, from the equator.*

SOLUTION.

Let, S, the ſun's place, be 18° 24′ of Taurus: then the arc ES will be 48° 24′; and therefore in the triangle SDE, right angled at D, we know the hypothenuſe, ES, and the angle DES $= 23° 28′ 30″ =$ the obliquity of the ecliptic; whence we ſhall get by the Table for right-angled triangles,

$$\textit{ſin.}\, DS = \frac{\textit{ſin.}\, ES \times \textit{ſin.}\, DES}{R}.$$

LOGARITHMICALLY.

9.873784 = *log. ſin.* 48° 24′
9.600263 = *log. ſin.* 23° 28′ 30″
9.474047 = *log. ſin.* 17° 19′ 49″ = DS.

PROBLEM II.

300. *Given the ſun's declination, as alſo the ſeaſon of the year; to determine his place in the ecliptic.*

SOLUTION.

Let us ſuppoſe the ſun's declination to be 17° 19′ 49″ northerly, and the ſeaſon to be the ſpring: then we muſt find the arc, ES, of the ecliptic contained between the ſun and firſt degree of Aries. Now to do this, it is manifeſt that, in the right-angled triangle DES, we ſhall know the angle E with its oppoſite ſide, DS; and therefore the hypothenuſe, ES, will be obtained by the common analogy, which gives, $\textit{ſin.}\, ES = \frac{\textit{ſin.}\, DS \times R}{\textit{ſin.}\, DES}$.

LOGA-

LOGARITHMICALLY.

9.474047 = *log. ſin.* DS = 17° 19′ 49″.
0.399737 = *arith. comp. ſin.* 23° 28′ 30″.
9.873784 = *log. ſin.* ES = 48° 24′ 0″.

SCHOLIUM.

301. From the preceding Problem the method of computing Tables for the ſun's motion in the ecliptic and his variations (by means of obſervations) will readily appear. For, the latitude of the place of the obſerver being accurately determined, and a quadrant fixed in the plane of the meridian, the obſervation of the ſun's meridian altitude gives his declination, or diſtance from the equator; from whence his longitude for any day is eaſily obtained, by the method above ſpecified.

PROBLEM III.

302. *Given the ſun's place in the ecliptic; to find his right aſcenſion, or* (which is the ſame) *the point,* D, *of the equator which paſſes over the meridian at the ſame time with the ſun.*

SOLUTION.

Suppoſing ſtill the arc ES to be 48° 24′, it is required to find the arc ED. Now the Table of right-angled triangles gives for this caſe, *tang.*

$$\text{ED} = \frac{\textit{tang.}\ \text{ES} \times \textit{coſ.}\ \text{DES}}{\text{R}}.$$

LOGARITHMICALLY.

10.051664 = *log. tang.* 48° 24′ 0″
9.962480 = *log. coſ.* 23° 28′ 30″
10.014144 = *log. tang.* 45° 55′ 58″

PROBLEM IV.

303. *Given the latitude of a place, and the ſun's declination; to find his ortive or occaſive amplitude, or at what diſtance from the true eaſt or weſt he riſes or ſets.*

SOLUTION.

Suppoſing always the latitude to be 48° 51′, and that the ſun's declination is 18° 30′ northerly; it is manifeſt that, all we have to do is to find the hypothenuſe of a right-angled triangle, DOS, wherein the angle DOS = the complement of the latitude, and the ſide DS = the ſun's declination, are given. We ſhall therefore have by the common analogy, $\textit{ſin.}\ OS = \frac{\textit{ſin.}\ DS \times R}{\textit{ſin.}\ DOS}$.

LOGARITHMICALLY.

9.501476 = *log. ſin.* DS = 18° 30′ 0″.
0.181758 = *arith. comp. log. ſin.* 41° 9′ 0″.
9.683229 = *log. ſin.* OS = 28° 49′ 45″.

COROLLARY I.

304. From this Problem it follows that, if the ſun's declination and amplitude be given, the latitude of the place will be eaſily obtained: for, according to this ſuppoſition, in the right-angled triangle DOS, we ſhall know (beſide the right angle) the hypothenuſe and leg DS; wherefore we ſhall have, $\textit{ſin.}\ DOS = \frac{\textit{ſin.}\ DS \times R}{\textit{ſin.}\ OS}$.

LOGARITHMICALLY.

9.501476 = *log. ſin.* DS = 18° 30′ 0″.
0.316771 = *arith. comp. log. ſin.* 28° 49′ 45″.
9.818247 = *log. ſin.* 41° 9′; *whence the lat.* = 48° 51′.

COROLLARY II.

305. In like manner, if we have the amplitude, OS, and latitude of a place, given, we ſhall eaſily find the ſun's declination correſponding to this amplitude; that is, $\textit{ſin.}\ DS = \frac{\textit{ſin.}\ OS \times \textit{ſin.}\ DOS}{R}$.

LOGARITHMICALLY.

9.683229 = *log. ſin.* 28° 49′ 45″.
9.818247 = *log. ſin.* 41° 9′ 0″.
9.501476 = *log. ſin.* 18° 30′ 0″.

PROBLEM V.

306. *Given the latitude of a place, and the ſun's place in the ecliptic; to find the point of the equator which comes to the horizon at the ſame time with the ſun, or* (which is the ſame) *the oblique aſcenſion correſponding to the ſaid latitude and place of the ſun.*

SOLUTION.

Let us ſuppoſe, as in the firſt Problem, that the ſun's place in the ecliptic is 18° 24′ of Taurus: then (by the ſame Problem) his declination will be 17° 19′ 49″; conſequently, in the right-angled triangle DOS, we ſhall know the leg DS with the angle DOS = the complement of the latitude, in order to find the leg OD. To do which, the general Table for right-angled triangles gives

$$\text{ſin. OD} = \frac{\text{tang. DS} \times \text{R}}{\text{tang. DOS}}$$

LOGARITHMICALLY.

9.494217 = *log. tang.* DS = 17° 19′ 49″.
0.058541 = *arith. comp. log. tang.* 41° 9′ 0″.
9.552758 = *log. ſin.* OD = 20° 55′ 15″.

Then, if from the right aſcenſion, DE, found in *art.* 302 } = 45° 55′ 58″
we ſubtract the arc thus found — — — 20° 55′ 15″
we ſhall have for the oblique aſcenſion, EO 25° 0′ 43″.

SCHOLIUM.

307. It is well known that at the equator the days and nights are conſtantly equal, becauſe in the right ſphere the points D and O coincide. In an oblique ſphere, the arc DO expreſſes the quantity which muſt be added to, or ſubtracted from, 90°,

in order to obtain half the length of the day at a place whoſe latitude is northerly, according as the ſun's declination is towards the north or ſouth.

It is manifeſt that, we have paid no regard to the augmentation cauſed by refraction.

PROBLEM VI.

308. *Given the latitude, and ſun's place in the ecliptic; to find the angle made by the ecliptic with the horizon at the inſtant of ſun-riſe.*

SOLUTION.

Let the ſun's place be ſtill ſuppoſed to be 18° 24′ of Taurus, the latitude 48° 51′, and conſequently its complement AOH or DOS 41° 9′: then (as already found in *art.* 299) the arc DS will be 17° 19′ 49″; whence, by making the calculation, as in *Prob.* IV. *ſin.* OS, the ortive amplitude will be found = 9.655800; alſo EO (obtained from the ſame *data* in the laſt Problem) = 25° 0′ 43″; and therefore, by the common analogy between the *ſines* of ſides and thoſe of their oppoſite angles, we ſhall get, $\text{ſin. ESO} = \frac{\text{ſin. SEO} \times \text{ſin. EO}}{\text{ſin. OS}}$.

LOGARITHMICALLY.

9.600263 = *log. ſin.* SEO = 23° 28′ 30″.
9.626141 = *log. ſin.* EO = 25° 0′ 43″.
0.344200 = *arith. comp. log. ſin.* OS.
9.570604 = *log. ſin.* ESO = 21° 50′ 32″.

SCHOLIUM.

309. By calculations, nearly ſimilar to thoſe made in the two preceding Problems, it would be eaſy to find the arc IE, of the ecliptic, contained between the meridian and interſection of the ecliptic and equator. For, having obtained the oblique aſcenſion EO, the arc AE, of the equator which is the complement thereof, will become known; whence, in the right-angled triangle IAE, the

ſide AE with the angle E equal to the obliquity of the ecliptic will be given.—We might likewiſe find the arc of the meridian intercepted between the equator and ecliptic, as well as the angle made by the ecliptic with the meridian, and the arc, IS, compriſed between the meridian and horizon.

We ſhall now ſubjoin a Problem whoſe ſolution depends only upon a common right-angled ſpherical triangle, though Mr. *Maupertuis* hath for that end made uſe of the differential (fluxional) calculation.

PROBLEM VII.

310. *The declination of a ſtar, and the latitude of a place being given; it is required to find,* 1°, *the moment when this ſtar appears to aſcend or deſcend perpendicularly to the horizon;* 2°, *the altitude of the ſtar in a vertical; and* 3°, *the angle which the ſaid vertical makes with the meridian.*

SOLUTION.

With a very little attention to the nature of the Problem it will appear, that the ſtar cannot poſſibly aſcend or deſcend perpendicularly to the horizon, unleſs the parallel, which it deſcribes, can be ſuppoſed to have ſome vertical touching it; and that the ſtar will be found in the point of contact of the ſaid parallel and vertical; yet that this thing can only prevail twice in a revolution of 24 hours. Moreover, it is eaſy to perceive, that no parallel can have any vertical to touch it, except its diſtance from the equator be greater than the latitude of the place; for provided it be leſs than, and on the ſame ſide of the equator with, the given latitude, it is very obvious that, in this caſe, all verticals whatever will cut it. This being premiſed, let ZST be a vertical touching the parallel GSM at the point S, the ſuppoſed place of a ſtar, whoſe declination hath for its complement the arc PS: then, on account of the point of contact of this vertical and parallel, it is manifeſt that, the arc PS is the leaſt that can be drawn from the point P to the vertical ZST: conſequently, the triangle PSZ will be right-angled at S, and we ſhall know therein (beſide the right angle) the ſide PS = the complement of the ſtar's declination, and the ſide PZ = the complement of the latitude; whence, if the difference between the right aſcenſion of the ſun and ſtar be known, the hour-angle ZPS will give the inſtant when this ſtar appears to aſcend perpendicularly to the horizon; whilſt the ſide ZS will give the complement of its altitude; and the angle PZS the poſition of the vertical required with reſpect to the meridian.——If then L be put for the latitude of the place; D for the declination of the ſtar; H′ for its altitude above the horizon, when it appears to aſcend vertically; H for the hour-angle ZPS; and V for the angle made by the vertical and meridian; the formulæ for right-angled triangles will give; $cof.\ H = \frac{cot.\ D \times R}{cot.\ L}$; $fin.\ H' = \frac{cof.\ L \times R}{cof.\ D}$, and $fin.\ V = \frac{cof.\ D \times R}{cof.\ L}$.——The applications to the Logarithms will be ſufficiently eaſy without examples.

PROBLEM VIII.

311. *Given the latitude of a place, and the ſun's declination; to find the time of his riſing and ſetting.*

SOLUTION.

Suppoſing the declination 18° 30′ northerly, and the latitude 48° 51′; it is manifeſt that in the right-angled triangle PSR, there will be given PR equal to the latitude, and PS the complement of the declination; and therefore for the angle RPS (the thing required) we ſhall have, $$coſ.\ RPS = \frac{tang.\ PR \times R}{tang.\ PS}.$$

LOGARITHMICALLY.

10.058541 = *log. tang.* PR = 48° 51′ 0″.
9.524520 = *arith. comp. log. tang.* 71° 30′ 0″.
9.583061 = *log. coſ.* RPS = 67° 29′ 17″.

Now, in order to know what hour of the morning this value of RPS correſponds to, we muſt have recourſe to the annexed Table; and we ſhall readily find, that

For	15°	reckon	1 *hour.*	on the day when the declination is 18° 30′ northerly, at a place whoſe latitude is 48° 51′, the ſun riſes 29 minutes 57 ſeconds after 4 o'clock.
	1°		4 *minutes.*	
	15′		1 *minute.*	
	1′		4 *ſeconds.*	
	15″		1 *ſecond.*	
	1″		4 *thirds.*	
	15‴		1 *third.*	

SCHOLIUM.

312. If the amplitude was known inſtead of the latitude, the angle RPS would be obtained by this proportion; *As the coſine of the declination is to the ſine total, ſo is the coſine of the amplitude to the ſine of the hour-angle,* RPS: and contrarily, if the time of ſun-riſe or ſun-ſet was given, the ortive or occaſive amplitude would be found by ſaying; *As the ſine total is to the coſine of the declination, ſo is the ſine of the hour-angle to the coſine of the amplitude.* ——We may likewiſe perceive that, if the ſun's declination and time of his riſing were given, the latitude of the place would be eaſily found; ſince, in the right-angled triangle SPR, we ſhould know the hypothenuſe and angle P.——We might alſo find the ſun's declination from the hour of his riſing and the latitude.——Moreover, this ſolution may be applied to the riſing or ſetting of a

planet or ſtar, provided we know its declination and right aſcenſion: for, if the ſun's right aſcenſion be found for the ſame day by means of his declination, the difference between it and that of the ſtar reduced into hours, minutes and ſeconds, by the foregoing Table, will give the time that the ſtar riſes before, or after, the ſun: by which means we ſhall be able to know at what time of the year a ſtar can be ſeen or not ſeen.——We may alſo make uſe of this Problem in order to find the ſun's horizontal refraction: for, ſuppoſing the true, and alſo apparent, time of his riſing to be known, if we take their difference, and reduce it into parts of the equator; then calculate the ſide ZS, and take 90° therefrom, the remainder will ſhew the quantity of his elevation occaſioned by refraction.

PROBLEM IX.

213. *Given the latitude of a place, the ſun's declination, and his altitude; to find the hour of the day.*

SOLUTION.

Fig. 35. Let the latitude be 48° 51′; the ſun's declination 18° 30′ northerly, and his altitude 52° 35′: then in the triangle PZS we ſhall know the three ſides, in order to find the angle ZPS: which will be eaſily done by the formulæ (*art.* 166), *ſin.* $\frac{1}{2}P = \frac{\sqrt{ſin.\ \frac{1}{2}s-b \times ſin.\ \frac{1}{2}s-c}}{\sqrt{ſin.\ b \times ſin.\ c}}$; wherein s denotes the ſum of the three ſides, and, b, c, the ſides adjacent to the angle required.——See the *Logarithmical* operation.

9.746247 = *log. ſin.* 33° 53'
8.789787 = *log. ſin.* 3° 32'
0.023043 = *arith. comp. log. ſin.* 71° 30'
0.181753 = *arith. comp. log. ſin.* 41° 9'

18.740830
9.370415 = *log. ſin.* ½ P = 13° 34' 14" ∴ P = 27° 8' 28"; the time corresponding to which is 10 hours 11 minutes 26 seconds and 8 thirds of the morning, if the angle ZPS be to the east of the meridian; or 48 minutes 33 seconds and 52 thirds after 1 in the afternoon, if it be to the weſt.

SCHOLIUM.

314. This Problem points out to us an eaſy method of finding the duration of twilight. For, if we conceive the ſun to be depreſſed 18° below the horizon (as the crepuſcular circle is uſually ſuppoſed to be), the arc ZS will be 108°: then if we calculate (as in the preſent Problem) the hour-angle ZPS, and afterwards ſubtract it from 180°, the remainder, reduced into parts of time, will give the beginning of twilight, or break of day. If it is the end of twilight which we would find, the angle ZPS itſelf muſt be reduced into parts of time.

OBSERVATION.

In the performance of this Problem, we muſt take care to correct the ſun's altitude by refraction and parallax (if material).——Tables of theſe corrections may be met with in most Treatises of Astronomy.

PROBLEM X.

315. *Given the latitude of a place, with the ſun's declination and altitude, to find the angle that the vertical, on which the ſun is, makes with the meridian.*

SOLUTION.

From the *data* it is manifeſt that, in the triangle PZS, the three ſides will be known, in order to find the angle PZS. Now this may be obtained by the method given in the laſt Problem; and therefore, if the parts given be ſuppoſed to have the ſame value in this that they had in that, the *Logarithmical* operation will ſtand thus:

9.746048 = *log. ſin.* ½ (s − b)
9.785597 = *log. ſin.* ½ (s − c)
0.181753 = *arith. comp. log. ſin.* 41° 2′
0.216377 = *arith. comp. log. ſin.* 35° 25′
19.929775
9.964887 = *log. ſin.* ½ PZS = 67° 18′ 3″.

whence PZS = 134° 36′ 6″; and conſequently the angle that the vertical, on which the ſun is, makes with the meridian 45° 23′ 54″.

SCHOLIUM.

316. From this operation it will be eaſy to find the four cardinal points; ſince, if with a line, repreſenting the vertical on which the ſun is, an angle be made equal to that above found, the meridian will be obtained. Now this method, provided we take care to correct the ſun's altitude by refraction, will be found very commodious for determining the meridian, as the changes in declination which prevail in corresponding altitudes are

hereby entirely avoided. It serves likewise to determine the variation of the compaſs.

PROBLEM XI.

317. *Given the latitude, ſun's declination, and hour of the day; to find the angle that the vertical, on which the ſun then is, makes with the meridian.*

SOLUTION.

Here it is manifeſt that, in the ſpherical triangle PZS the two ſides, PZ, PS, with their included angle, ZPS, are known. Now let us ſuppoſe the declination to be 18° 30′, and that the ſun was on the vertical, whoſe poſition is required, at 1 [illegible] minutes, 26 ſeconds paſt noon, then, by the Table given in *Prob.* VIII. the hour angle correſponding to this time will be 27° 8′ 30″; the arc PS will be ſtill 71° 30′, and PZ 41° 9′. This being premiſed, in order to find the angle PZS, let fall the arc SR perpendicularly upon the ſide PZ from the vertex of the (third) angle S (as required in *Table* II. for the reſolution of oblique-angled ſpherical triangles), and we ſhall in the firſt place get, for the part there called I. *ſeg. tang.* PR $= \frac{\text{coſ. } ZPS \times \text{tang. } PS}{R}$; or,

LOGARITHMICALLY,

9.949332 = *log. coſ.* 27° 8′ 30″
10.475480 = *log. tang.* 71° 30′ 0″
10.424812 = *log. tang.* 69° 23′ 37″ = PR;

then the ſecond ſegment, RZ, will be 28° 14′ 37″; and therefore, by means of the ſame Table, we ſhall laſtly have, *tang.* PZS $= \frac{\text{tang. } ZPS \times \text{ſin. } PR}{\text{ſin. } RZ} =$ 45° 23′ 52″. — See the *Logarithmical* operation.

9.709819 = *log. tang.* ZPS = 27° 8′ 30″.
9.971285 = *log. ſin.* PR = 69° 23′ 37″.
0.324936 = *arith. comp. log. ſin.* 28° 14′ 7″
10.006040 = *log. tang.* PZS = 45° 23′ 53″ or 134° 36′ 7″.

SCHOLIUM.

318. This Problem may ſerve for finding the declination of any vertical plane. — We may likewiſe apply it in order to determine the poſition of an avenue or well-built ſtreet, by obſerving the moment when the ſun's ſhadow paſſes by the bottom of the trees or walls thereof. —— Moreover, if the ſun's altitude above the horizon be likewiſe known at the given time, the preceding calculation will be rendered conſiderably more ſimple, and reduced to this analogy; *As the coſine of the ſun's altitude is to the ſine of the hour-angle, ſo is the coſine of the declination to the ſine of the angle made by the vertical and meridian.*

PROBLEM XII.

319. *Given the hour of the day, with the ſun's declination and altitude; to find the latitude of the place.*

SOLUTION.

Fig. 35. In this Problem, it is manifeſt that, the ſide PS = the complement of the ſun's declination; the ſide ZS = the complement of his altitude, and the angle P, are given, to find the ſide ZP. Now let us ſuppoſe, as in the preceding Problems, that the ſun's declination is 18° 30′, that his altitude is 52° 35′, and conſequently the arc ZS = 37° 25′; as alſo that the time is 11 minutes 26 ſeconds paſt 10, and of courſe the angle P = 27° 8′ 30″; then, letting fall the perpendicular SR from the angle S, PR will, in the firſt place, be determined by the formula, in the general Table for oblique-angled triangles, *tang.* I. *ſeg.* = *coſ. giv. ang.* × *tang. of its adj. ſide*: in the next place, we ſhall have,

coſ.

cof. II. ſeg. = cof. RZ = $\frac{\text{cof. I. ſeg.} \times \text{ſin. } 52^\circ\ 35'}{\text{ſin. } 18^\circ\ 30'}$; from whence the value of the ſide PZ = the complement of the latitude will be immediately deduced.——See the operation at length.

9.949332 = *log. cof.* 27° 8′ 30″
10.475480 = *log. tang.* 71° 30′ 0″
10.424812 = *log. tang.* 69° 23′ 37″ = PR.

2°. 9.546475 = *log. cof.* - - - 69° 23′ 37″.
9.899951 = *log. ſin.* - - - - 52° 35′ 0″.
0.498524 = *arith. comp. log. ſin.* 18° 30′ 0″.
9.944950 = *log. cof.* RZ = 28° 14′ 37″.

then, ſubtracting the ſecond ſegment = 28° 14′ 37″ from the firſt = 69° 23′ 37″, we ſhall get 41° 9′ for the complement of the latitude; and conſequently, 48° 51′ for the latitude itſelf.

PROBLEM XIII.

320. *Given the ſun's declination, his altitude, and the azimuth, or angle made with the meridian by the vertical on which the ſun is found at the moment of obſervation; to determine the latitude of the place.*

SOLUTION.

It is evident that, in the triangle PZS the ſides, PS, ZS, with the angle PZS oppoſite to the ſide PS, are given: therefore, in order to obtain the ſide PZ, we muſt (according to the Table for oblique angled ſpherical triangles) let fall from the vertex of the angle S an arc, SR, as perpendicular upon the unknown ſide; and we ſhall get *tang.* RZ = *cof.* PZS × *tang.* ZS, and *cof.* PR = $\frac{\text{cof. RZ} \times \text{cof. PS}}{\text{cof. ZS}}$: the values whereof, ſuppoſing the ſun's declination to be 18° 30′; his altitude above the horizon 52° 35′; and the angle PZS, made by the vertical with the meridian 134° 36′ 7″, will be obtained *Logarithmically*, as follows:

1°.

1°. 9.846448 = *log. coſ.* 134° 36′ 7″.
9.883672 = *log. tang.* 37° 25′ 0″.
9.730120 = *log. tang.* 28° 14′ 37″ = RZ.

2°. 9.944948 = *log. coſ.* - - - - - - 28° 14′ 37″.
9.501476 = *log. ſin.* - - - - - - 18° 30′ 0″.
0.100049 = *arith. comp. log. ſin.* 52° 35′ 0″.
9.546473 = *log. coſ.* RP = 69° 23′ 37″.

then, ſubtracting 28° 14′ 37″ from 69° 23′ 37″, we ſhall find the ſide PZ = the complement of the latitude to be 41° 9′; whence the latitude itſelf is 48° 51′.

SCHOLIUM.

321. From what hath been ſaid it may be readily collected, that if any three of theſe ſix things, the latitude, the ſun's altitude, his declination, the hour-angle, azimuth and angle at the ſun, be given at pleaſure, the other three will neceſſarily become known by the rules of Trigonometry: for which reaſon we hope to be excuſed for not particularizing all the different caſes that reſult from their combinations.

PROBLEM XIV.

322. *The longitude and latitude of a ſtar being given; to find its declination.*

SOLUTION.

Suppoſe the longitude of the ſtar to be 198° 27′, and its latitude 31° 2′ northerly: then, it is evident that, in the triangle PKS, we know the ſide PK = to the obliquity of the ecliptic = 23° 28′ 30″; the ſide KS equal to the complement of the latitude = 58° 58′, and the angle PKS = to the longitude of the ſtar leſs 90°, in order to find the ſide PS, the complement of the declination required. Now to do this, let fall (as the general Table for oblique-angled triangles requires) from one of the unknown angles, S, a perpendicular arc SR, upon he oppoſite ſide PK (which will fall without the angle S; becauſe the angle PKS is obtuſe, and give the arc KR for the part denoted by *ſeg.* I. and PK + KR or PR for *ſeg.* II.), ſo will; 1°, *tang.* KR $= \frac{\textit{coſ.}\ \text{PKS} \times \textit{tang.}\ \text{KS}}{\text{Rad.}}$, and 2°, *coſ.* PS or *ſin.* DS $= \frac{\textit{coſ.}\ \text{KS} \times \textit{coſ.}\ \text{PR}}{\textit{coſ.}\ \text{KR}}$.——See the *Logarithmical* operation.

1°. 9.500342 = *log. coſ.* PKS = 108° 27′ 0″.
10.220654 = *log. tang.* KS = 58° 58′ 0″.
9.720996 = *log. tang.* KR = 27° 44′ 41″.
∴ PR = 51° 13′ 11″.

2°. 9.712260 = *log. coſ.* KS.
9.796804 = *log. coſ.* PR.
0.054042 = *arith. comp. log. coſ.* KR.
9.563106 = *log. ſin.* 21° 26′ 58″ = DS = *dec. required.*

PROBLEM XVII.

823. *Suppoſing the ſame things given as in the preceding Problem; it is required to find the right aſcenſion of the ſame ſtar, as alſo the angle made by a meridian and the circle of longitude paſſing through this ſtar.*

SOLUTION.

From the *data* it will be no difficult matter to perceive, that in the triangle PKS two ſides and their included angle are ſtill known; and likewiſe that the ſtar's right-aſcenſion will be determined by ſubtracting the angle KPS from 270°. Now, in order to obtain the value of this angle, together with that of the angle at the ſtar at the ſame time, it will be neceſſary to make uſe of ſome of the analogies of *Neper*; and thoſe that appear to conduce to this end will be found by *Tab.* III. to be the two following: Fig. 36.

1°, *As the ſine of half the ſum of the ſides,* PK *and* KS, *is to that of half their difference, ſo is the cotangent of half their included angle,* PKS, *to the tangent of half the difference of the angles above the baſe*; and, 2°, *as the coſine of half the ſum of theſe ſides is to the coſine of half their difference, ſo is the cotangent of half* *their included angle to the tangent of half the ſum of the angles above the baſe.* — Here follows the *Logarithmical* operation.

58° 58′ 0″ = KS. | 58° 58′ 0″.
23 28 30 = PK. | 23 28 30.

82° 26′ 30″ = KS + PK | 35° 29′ 30″.
$41^\circ\ 13'\ 15'' = \frac{KS + PK}{2}$ | $17^\circ\ 44'\ 45'' = \frac{KS - PK}{2}$

108° 27′ 0″ = PKS.

$\frac{PKS}{2} = 54^\circ\ 13'\ 30''$, *whoſe comp.* = 35° 46′ 30″.

9.484008 = *log. ſin.* 17 44 45	9.978827 = *log. coſ.* 17° 44′ 45″
9.857670 = *log. tang.* 35 46 30	9.857670 = *log. tan.* 35° 46′ 30″
0.181139 = *ac. l. ſin.* 41 13 15	0.123682 = *a. c. l. coſ.* 41° 13′ 15″
9.522817 = *log. tang.* 18° 25′ 56″	9.960179 = *log. tan.* 42° 22′ 36″

Then, as half the ſum of the angles at P and S is 42° 22′ 36″, and half their difference 18° 25′ 56″, the angle KPS will be 60° 48′ 32″, and of courſe the ſtar's right aſcenſion 209° 11′ 28″; and the angle at the ſtar 23° 56′ 40″.

SCHOLIUM.

324. The right aſcenſions and declinations of the ſtars, as likewiſe their longitudes and latitudes, are of exceeding great and frequent uſe in Aſtronomy. By their means the reſpective poſitions and ſituations of the ſtars are aſcertained and determined, and from thence ſuch catalogues thereof formed and deduced, as are neceſſary for comparing the motions of the Planets, as well as of the Comets that may at different times be obſerved to appear. However, the preference is uſually given to the longitudes and latitudes; becauſe, it hath been for a long time believed that theſe elements undergo no change, particularly, if care be taken to reckon the longitudes from ſome *fixed* ſtar; whilſt the preceſſion of the equinoxes occaſions a conſiderable change in the right aſcenſions and declinations. But abating all conſiderations of this nature, it is eaſy to perceive that, the ſeveral elements of the Theory of the ſtars may be deduced from one another,

other, by a very great variety of combinations; to particularize which in this place would be foreign to our deſign. The right aſcenſions and declinations of the ſtars may be uſed, as well as thoſe of the ſun, to determine the latitudes of places on the earth: from thence their ortive or occaſive amplitudes may be deduced, and of conſequence thoſe of ſuch ſtars as may have the ſame poſitions in the ſphere: from thence it may be found, what ſtars never ſet in a given latitude; likewiſe, by their means it may be known, how long a ſtar tranſits the meridian before, or after, the ſun; and of conſequence when the heliacal riſing or ſetting of a ſtar commences, that is, at what time of the year the ſun's rays render it viſible or inviſible. The moſt ſimple methods, that have been adopted for determining the poſitions of the ſtars, conſiſt in obſerving their meridian altitudes and the inſtant of their tranſiting the meridian; as by that means their right aſcenſions and declinations are immediately obtained, and from thence their longitudes and latitudes, by the converſe of the two Problems laſt ſolved.

PROBLEM XVI.

325. *Given the ſun's place in the ecliptic, the hour of the day or night, and the latitude of the place, alſo the altitude of a ſtar, and the angle which the vertical, on which it is obſerved, makes with the meridian; to determine the declination of this ſtar, and its right aſcenſion.*

SOLUTION.

Let us ſuppoſe the ſun's place to be 18° 24′ of Taurus; the time 20 minutes paſt 9 in the evening, and the latitude of the place 48° 51′; alſo the altitude of the ſtar, in the vertical circle on which it is obſerved, to be 43° 15′, and the azimuth 49° 24′ from the ſouth: then, it is manifeſt that in the triangle PZS, we ſhall know the two ſides, PZ, ZS, with their included angle, to find the arc *Fig.* 35.

DS which repreſents the declination of the ſtar, and the diſtance of the point D from the firſt point of Aries. Now, to obtain the declination (which will be done by means of the ſide PS), let fall from the vertex of the angle S a perpendicular arc, SR, upon the ſide PZ, and the Table for oblique-angled triangles will give theſe two formulæ; 1°, *tang.* RZ = *coſ.* PZS × *tang.* ZS, and 2°, *coſ.* PS or *ſin.* DS = $\frac{\textit{coſ.}\ ZS \times \textit{coſ.}\ PR}{\textit{coſ.}\ RZ}$; whoſe values will be found *logarithmically*, as follows:

9.830509 = *log. coſ.* PZS
10.026546 = *log. tang.* ZS
9.857055 = *log. tang.* RZ = 35° 44′ 11″

[illegible] = *log. coſ.* ZS
[illegible] = *log. coſ.* PR
0.090598 = *ar. comp. log. coſ.* RZ
9.520305 = *log. ſin.* 19° 21′ 5″ = DS.

The ſtar's declination being thus diſcovered to be equal 19° 21′ 5″, the next thing to be done is to find its right aſcenſion. In order to which, we muſt in the firſt place determine the hour-angle ZPS by the common analogy between the *ſines* of ſides and thoſe of their oppoſite angles, which gives, *ſin.* ZPS = $\frac{\textit{ſin.}\ PZS \times \textit{ſin.}\ ZS}{\textit{ſin.}\ PS}$ = *ſin.* 34° 37′ 43″, as the ſubſequent operation will evince:

9.866935 = *log. ſin.* PZS = 132° 36′ 0″
9.862353 = *log. ſin.* ZS = 46° 45′ 9″
0.025257 = *arith. comp. log. ſin.* 70° 38′ 55″
9.754545 = *log. ſin.* ZPS = 34° 37′ 43″:

then adding this angle, the ſun's diſtance from the meridian = 140° (found by reducing the given time 9 hours 20 minutes into parts of the equator), and his right aſcenſion = 45° 55′ 58″ (*art.* 302), into one ſum, we ſhall get 220° 33′ 41″ for the ſtar's right aſcenſion ſought.——If we would know the time of the ſtar's paſſing over the meridian, we need only reduce 34° 37′ 43″ into parts of time, add the reſult, 2 hours 18 minutes 31 ſeconds, to 9 hours 20 minutes, and we ſhall find 11 hours 38 minutes 31 ſeconds in the evening for the time required.

PRO-

PROBLEM XVI.

326. *The right aſcenſions and declinations of two fixed ſtars, obſerved on the ſame vertical given in poſition, together with their diſtance or the arc of this vertical contained between them, being known; it is required to determine the latitude of the place of obſervation.*

SOLUTION.

From the *data* it will manifeſtly appear, that the three ſides of the triangle PS*s* are given, as well as the angle at P; ſince the arcs, PS, P*s*, are the complements of the declinations of the two ſtars (known by the hypotheſis), S*s* their diſtance on the vertical given in poſition, and the angle SP*s* the meaſure of the difference of their right aſcenſions: wherefore, we ſhall have, *ſin.* S*s* : *ſin.* SP*s* :: *ſin.* *s*P : *ſin.* PSZ $= \frac{\textit{ſin.}\, sP \times \textit{ſin.}\, SPs}{\textit{ſin.}\, Ss}$. By this means two angles, and a ſide oppoſite to one of them, will be actually known in the triangle PZS, viz. the azimuth or angle PZS, the angle PSZ, and the ſide PS: whence we ſhall have, *ſin.* PZS : *ſin.* PSZ :: *ſin.* PS : *ſin.* PZ, or, by ſubſtituting for *ſin.* PSZ its preceding value and taking away the fraction, *ſin.*PZS × *ſin.*S*s* : *ſin.* *s*P × *ſin.*SP*s* :: *ſin.*PS : *ſin.*PZ. Fig. 37.

Now, in order to apply this ſolution to a particular example, ſuppoſe the diſtance S*s* to be = 28° 30′; the right aſcenſion of S = 78° 24′, and its declination = 27° 25′; the right aſcenſion of *s* = 104° 52′, and its declination = 12° 18′; and laſtly the azimuth AZK = 73° 36′: then will the angle SP*s* be = 26° 28′; the arc PS = 62° 35′, and the arc P*s* = 77° 42′. This being premiſed, the value of PZ will be eaſily obtained, *Logarithmically*, as follows:

9.948257 = *log. ſin.* - - - - - 62° 35′ 0″ = PS.
9.989915 = *log. ſin.* - - - - - 77° 42′ 0″ = P*s*.
9.649020 = *log. ſin.* - - - - - 26° 28′ 0″ = SP*s*.
0.018039 = *arith. comp. log. ſin.* 73° 36′ 0″ = PZS.
0.321337 = *arith. comp. log. ſin.* 28° 30′ 0″ = S*s*.
9.926568 = *log. ſin.* - - - - - 57° 36′ 43″ = *comp. of lat.*

327.

COROLLARY I.

327. It is very evident, that the hour-angle ZPS may be alſo found; and therefore, if the ſun's place and his right aſcenſion be known, the difference between it and that of the ſtar, S, will become known: then, if from this difference the arc AD be taken, the remainder will give the arc of the equator contained betwixt the meridian of the place, and that paſſing through the ſun; from whence the moment of obſervation will be eaſily deduced.——It is no leſs evident, that we may likewiſe find, by purſuing the calculation, the altitude of each of the ſtars at the inſtant of obſervation.

COROLLARY II.

328. If, inſtead of the azimuth of the two ſtars, the altitude of one of them was given, it will be readily perceived that, by a calculation nearly ſimilar, the elevation of the pole might be obtained: for we ſhould ſtill know the three ſides of the triangle PSs with the angle P, and thence the angle PSZ contained between the given ſides, PS, ZS, of the triangle, PSZ.

COROLLARY III.

329. If, beſides the declinations and right aſcenſions of the ſtars obſerved on the ſame vertical, the hour-angle ZPS was given, we ſhould again be able to obtain the latitude of the place; ſince, in the triangle PSZ, we ſhould then have the angles, ZPS, ZSP, above the ſide SP, known by ſuppoſition.

COROLLARY IV.

330. Laſtly, if the diſtance and declinations of the ſtars, together with the altitude of one of them, were given, every thing beſides might be found as in the Problem itſelf. And hence we may perceive, how uſeful a thing it is to be furniſhed with accurate Tables of the declinations, right aſcenſions,

ſions, diſtances, longitudes and latitudes, of the ſtars, as from thence the latitudes of places may be found with the greateſt imaginable facility; nay, even without the neceſſity of uſing any large inſtruments, a plumb-line being ſufficient for obſerving whether two ſtars be ſituate on the ſame vertical or not.

PROBLEM XVIII.

331. *Given the longitudes and latitudes of two places upon the terreſtrial globe; to find their itinerary diſtance, or* (which is the ſame) *the arc of the great circle contained between them.*

SOLUTION.

Let the points, S, *s*, denote the propoſed places: then, it is manifeſt that, in the triangle SP*s* we ſhall know the two ſides, SP, *s*P, the complements of their latitudes, together with the angle P = the difference of their longitudes; whence the ſide S*s* will be found by the third caſe of the Table for oblique angled ſpherical triangles; after which we need only reduce the reſulting number of degrees, minutes and ſeconds, into leagues, by reckoning 25 to a degree, and the thing required will be effected. *Fig.* 27.

PROBLEM XIX.

332. *Given the latitudes and diſtance of two cities; to find their difference in longitude.*

SOLUTION.

The firſt thing to be done, in this Problem, is to reduce the given diſtance into degrees, minutes and ſeconds, of a great circle: then we ſhall have a ſpherical triangle, SP*s*, whereof the three ſides will be known; from whence the angle P will be eaſily found, which is the meaſure of the difference in longitude ſought.

COROLLARY.

333. Hence, if the poſition of one of the two Cities be determined, that of the other will be de-

termined likewiſe; and with ſo much the greater exactneſs, as the given diſtance ſhall be meaſured or given with the greater preciſion. And contrarily, if the difference of the longitudes and the diſtance of two Cities be given, together with the latitude of one of them, the latitude of the other City will be alſo had: but we ſeldom make uſe of this method, becauſe there is an infinite variety of better methods for finding the latitude, as we may be readily convinced from what hath been previouſly ſaid.

SCHOLIUM.

334. The preceding Problems, containing nearly all the caſes of right or oblique-angled ſpherical triangles, are ſufficient for pointing out the manner of obtaining the *Numerical* ſolutions of the different Problems of Trigonometry, by means of the formulæ demonſtrated in the foregoing Chapters. In this Chapter we had nothing farther in view, than the application of a few pertinent examples to the *Logarithms*; for which reaſon we have omitted taking notice of the different modifications whereof the things given might be ſuſceptible, with reſpect to parallax, refraction, the aberration of light, or nutation of the earth's axis. Theſe Theories can, properly ſpeaking, belong only to a complete Treatiſe of Aſtronomy: and therefore, we flatter ourſelves that it will be deemed enough for us to have laid down methods in this Work of ſolving the ſeveral trigonometrical caſes, that may chance to occur in the ſtudy of this ſcience; the difficulties whereof will riſe in proportion to that rigor and exactneſs, wherewith we may be deſirous of calculating the various *phænomena* therein conſidered.

FINIS.

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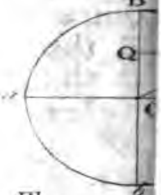

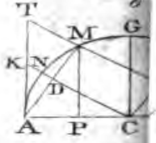

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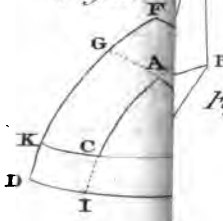

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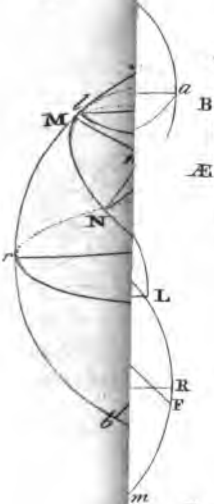

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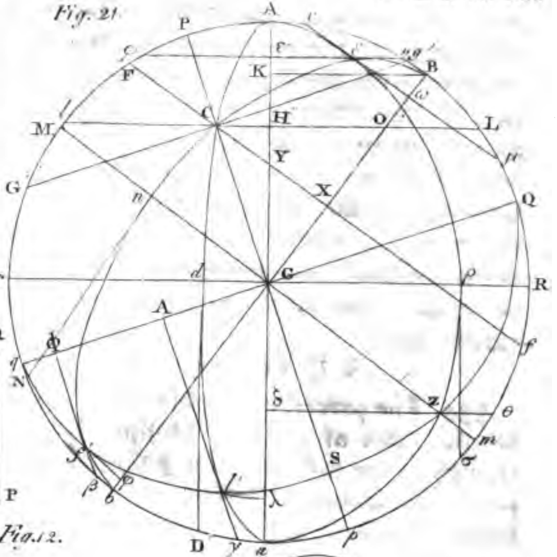

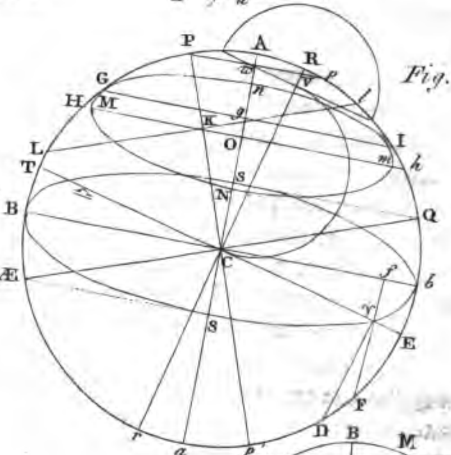

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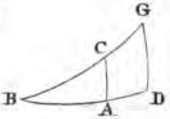

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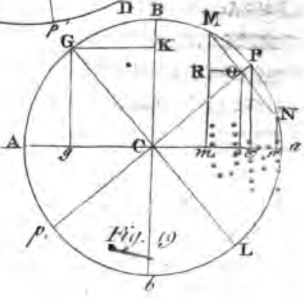

Fig. 19

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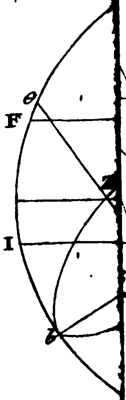
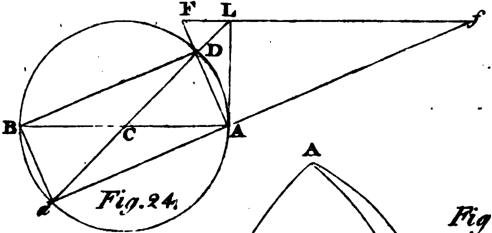
Pl. II at the End.

Fig. 2





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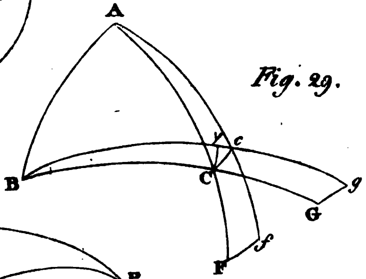

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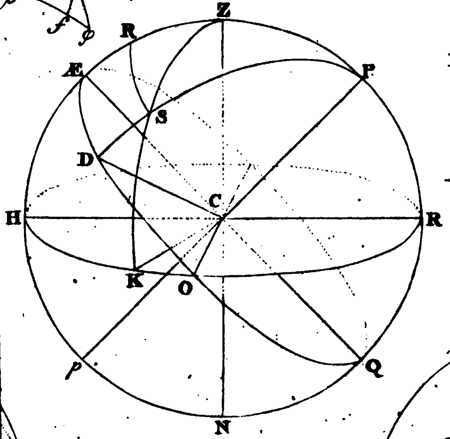
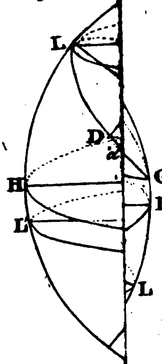
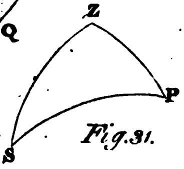


Fig. 35.

Fig. 32.





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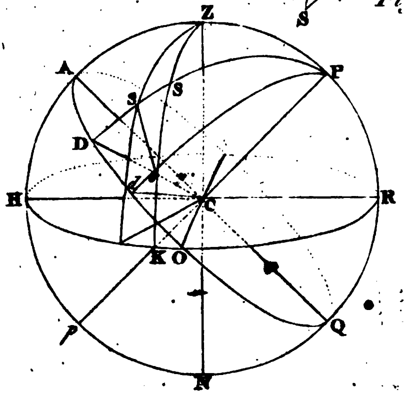

Fig. 37.

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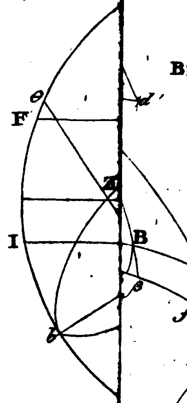
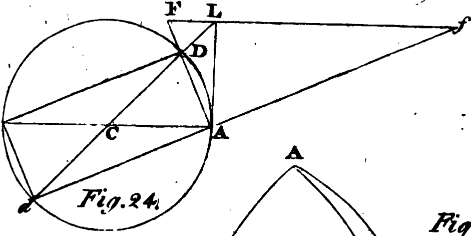
Fig. 22.





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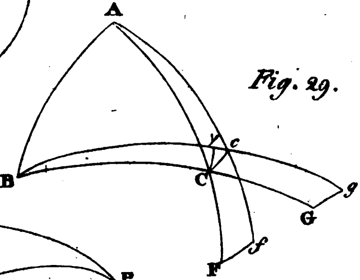

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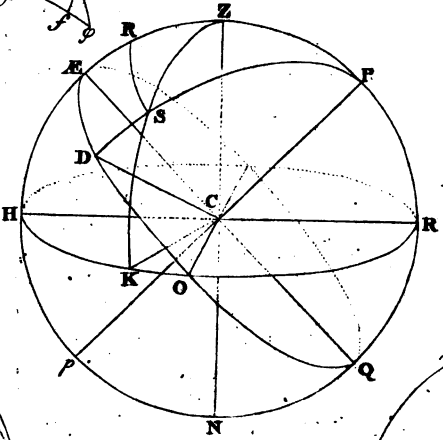

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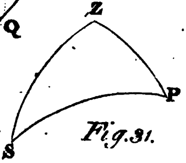

Fig. 31.

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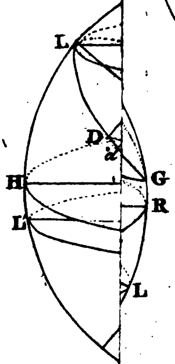

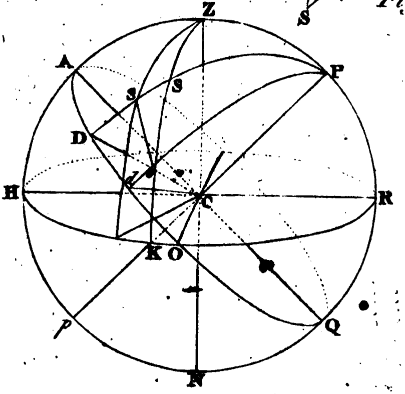

Fig. 37.

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